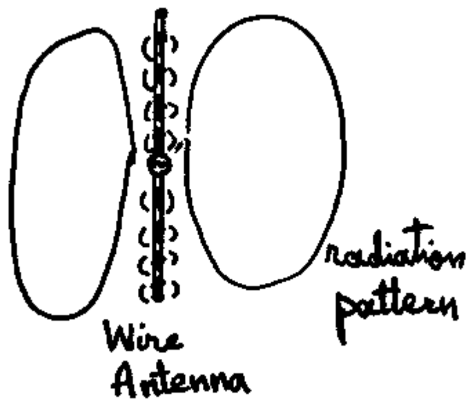
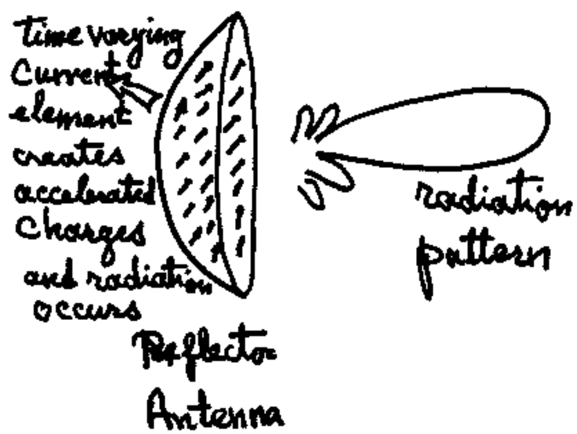
What is the BIG question?

Given the distribution of time varying current, find the radiation characteristics of the antenna.





The overall antenna radiation performance can be obtained by the superposition (in vector sense) of the elementary current elements.

An elegant way to solve radiation problem from Maxwell's equations (M.E.)

Time harmonic ($e^{j\omega t}$)
M.E. in homogeneous and isotropic medium

①
$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$
$$\nabla\times\vec{H} = j\omega\epsilon\vec{E} + \vec{J}$$
$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon}$$
$$\nabla\cdot\vec{B} = 0$$
$$\nabla\cdot\vec{J} = -j\omega\rho$$

this is known
find $\vec{E}$ & $\vec{H}$

②
$$\vec{B} = \nabla\times\vec{A} \Rightarrow \vec{H} = \frac{1}{\mu}\vec{B} = \frac{1}{\mu}\nabla\times\vec{A}$$

Vector potential yet to be determined

observe that
$$\nabla\cdot\nabla\times A = 0$$
vector Calculus identity

Combine

$$\nabla\times(\vec{E}+j\omega\vec{A}) = 0$$
then
$$\vec{E}+j\omega\vec{A} = -\nabla\Phi \quad ③$$
observe
$$\nabla\times\nabla\Phi = 0$$
Vector Calculus identity

Scalar potential yet to be determined

An elegant way to solve radiation problem from Maxwell's equations

② $\vec{H} = \frac{1}{\mu}\nabla\times\vec{A}$

M.E. $\nabla\times\vec{H} = j\omega\epsilon\vec{E} + \vec{J}$

$\Rightarrow \frac{1}{\mu}\nabla\times\nabla\times\vec{A} = j\omega\epsilon\vec{E} + \vec{J}$, with $\vec{E} = -j\omega\vec{A} - \nabla\Phi$ ③

$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

vector calculus identity

then

$$\boxed{\nabla^2\vec{A} + \omega^2\mu\epsilon\vec{A} - \nabla(j\omega\mu\epsilon\Phi + \nabla\cdot\vec{A}) = -\mu\vec{J}} \quad ④$$

③ $\vec{E} = -j\omega\vec{A} - \nabla\Phi$

M.E. $\nabla\cdot\vec{E} = \frac{\rho}{\epsilon}$

$\Rightarrow \nabla\cdot(-j\omega\vec{A} - \nabla\Phi) = \frac{\rho}{\epsilon}$

$\nabla^2\Phi \Leftarrow \nabla\cdot\nabla\Phi = \nabla^2\Phi$

Vector calculus identity

then

$$\boxed{\nabla^2\Phi + j\omega\nabla\cdot\vec{A} = -\frac{\rho}{\epsilon}} \quad ⑤$$

Note: Equations ④ and ⑤ are coupled. ☹

Lorentz Condition (or Gauge)

To uncouple previous equations Lorentz condition is introduced:

$$\nabla \cdot \vec{A} + j\omega\mu\epsilon\,\Phi = 0$$

then

$$\boxed{\begin{aligned} \nabla^2 \vec{A} + \beta^2 \vec{A} &= -\mu \vec{J} \\ \nabla^2 \Phi + \beta^2 \Phi &= -\frac{1}{\epsilon}\rho \end{aligned}}$$

$\beta = \omega\sqrt{\mu\epsilon}$ propagation constant

these are vector and scalar wave equations.

Observation: these differential equations relate the electromagnetic potentials to sources.

Solution of the Wave Equation

Vector Wave eq. for vector potential

$$\underbrace{\nabla^2 \vec{A} + \beta^2 \vec{A}}_{\text{this a vector}} = -\mu \vec{J} \Rightarrow \text{this is a vector}$$

In Cartesian Coordinates:

$$\underbrace{\nabla^2(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)}_{(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}} + \beta^2(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) = -\mu(J_x\hat{x} + J_y\hat{y} + J_z\hat{z})$$

$(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z} \Rightarrow$ Only true in Cartesian coordinates

Equating each component:

$$\left.\begin{aligned} \nabla^2 A_x + \beta^2 A_x &= -\mu J_x \\ \nabla^2 A_y + \beta^2 A_y &= -\mu J_y \\ \nabla^2 A_z + \beta^2 A_z &= -\mu J_z \end{aligned}\right\} \text{Three scalar wave equations}$$

Question: How to solve scalar wave eq. in unbounded space.

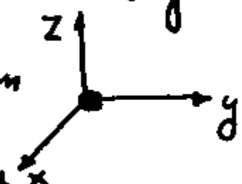
Solution of Scalar Wave eq. in Unbounded Space
Source at the Origin (1)

Let us first seek solution for the source at Origin :

$$\nabla^2 g(\vec{r}) + \beta^2 g(\vec{r}) = -\delta(\vec{r})$$

delta scalar source at the origin

delta function



Due to the obvious spherical symmetry : $g(\vec{r}) = g(r,\theta,\phi) = g(r)$ does not depend on θ and ϕ

then :

$$\nabla^2 g(r) + \beta^2 g(r) = -\delta(r)$$

Scalar Laplacian in Spherical Coordinates :

$$\nabla^2 g = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} g\right) + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 g}{\partial \phi^2}}_{\text{Zero in this case}}$$

then :

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g}{\partial r}\right) + \beta^2 g(r) = -\delta(r)}$$

Solution of Scalar Wave eq.
Source at the origin (2)

From partial differential eq. to ordinary differential eq.

partial derivatives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g}{\partial r}\right)+\beta^2 g(r)=-\delta(r)$$

Since $g(r)$ is only a function of r (i.e. not $\vec{r}$) then : $\frac{\partial g(r)}{\partial r}=\frac{d\,g(r)}{d r}$ ordinary derivative

Finally: $$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg}{dr}\right)+\beta^2 g(r)=-\delta(r)}$$

$$\frac{1}{r^2}\left[2r\frac{dg}{dr}+r^2\frac{d^2g}{dr^2}\right]$$

This is a second order linear ordinary differential eq. with variable coefficients

Solution of Scalar Wave eq.

Source at the origin (3)

Recall: $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg(r)}{dr}\right)+\beta^2 g(r)=-\delta(r)$$

This second order differential eq. can admit two solutions : $\dfrac{e^{-j\beta r}}{4\pi r}$ or $\dfrac{e^{+j\beta r}}{4\pi r}$

these two solutions are linearly independent

Which one should we use ? : The one which gives outgoing wave for $e^{j\omega t}$ time dependence.

outgoing and incoming waves

Physical quantity

$$\underbrace{\mathrm{Re}}_{\text{real part}}\left(e^{j\omega t}\,\frac{e^{\pm j\beta r}}{4\pi r}\right)=\underbrace{\frac{1}{4\pi r}}_{\substack{\text{Wave amplitude}\\ \text{decays as } \frac{1}{r}}}\cos(\omega t\pm\beta r)$$

$\omega \Rightarrow 2\pi f$ ← frequency

$\beta \Rightarrow \omega\sqrt{\mu\epsilon}$

Note : $\cos(\omega t-\beta r)$ characterizes an outgoing wave

wave front : $\omega t-\beta r=\text{Const.}\ \Rightarrow\ \dfrac{dr}{dt}=\text{velocity}=\dfrac{\omega}{\beta}=\dfrac{1}{\sqrt{\mu\epsilon}} \Rightarrow c$ speed of the light

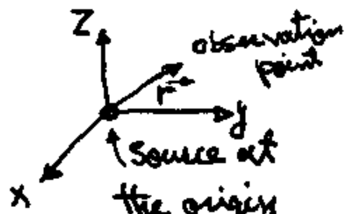
$$\boxed{\underbrace{\lambda}_{\text{wave length}}=c\underbrace{T}_{\text{Period}}=\frac{c}{f}\Rightarrow\beta=\frac{2\pi}{\lambda}}$$

Note : $\cos(\omega t+\beta r)$ Characterizes an incoming wave

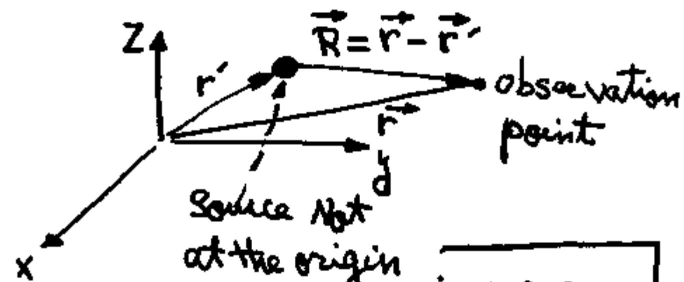
$\boxed{\text{We use outgoing waves for unbounded space}}$

Solution of scalar wave eq.

Source not at the origin



$$g(r) = \frac{e^{-j\beta r}}{4\pi r}$$



$$\boxed{g(\vec{r}, \vec{r}') = \frac{e^{-j\beta|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}}$$

Note: $R = |\vec{r} - \vec{r}'|$ length of vector $\vec{R}$

therefore $$\boxed{\nabla^2 g(\vec{r}, \vec{r}') + \beta^2 g(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}')}$$

Note: $|\vec{r} - \vec{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

Solution of the Vector wave eq. for Vector potential $\vec{A}$

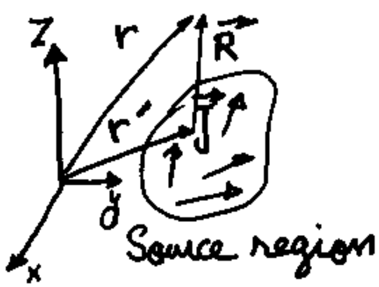
Recal: $\nabla^2 \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} + \beta^2 \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = - \begin{Bmatrix} \mu J_x \\ \mu J_y \\ \mu J_z \end{Bmatrix}$ three scalar wave equations

Recal: $\nabla^2 g(\vec{r},\vec{r}\,') + \beta^2 g(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$ ⟹ usualy called Green's function

By the argument of the superposition $\Longrightarrow$ $\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = \int \begin{Bmatrix} \mu J_x \\ \mu J_y \\ \mu J_z \end{Bmatrix} \dfrac{e^{-j\beta R}}{4\pi R}\, dv'$

$R \Rightarrow |\vec{r}-\vec{r}\,'|$

⟹ over the source region



Or in a vector form: $\boxed{\vec{A} = \int_{\substack{\text{Source} \\ \text{region}}} \vec{J}(\vec{r}\,') \dfrac{e^{-j\beta|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\, dv'}$

From vector Potential $\vec{A}$ to $\vec{E}$ & $\vec{H}$ fields

Recall: $$\vec{H}(\vec{r}) = \frac{1}{\mu} \nabla \times \vec{A}(\vec{r})$$

from M.E. $$\vec{E}(\vec{r}) = \frac{1}{j\omega\epsilon} [\nabla \times \vec{H}(\vec{r}) - \vec{J}(\vec{r})]$$

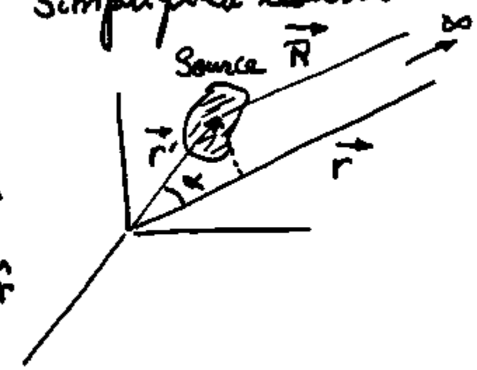
only exist at the source region

Recall: $$\vec{A}(\vec{r}) = \int_{\substack{\text{over} \\ \text{source region}}} \vec{J}(\vec{r}') \frac{e^{-j\beta|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} dv'$$

Observation: This is the answer to our BIG question.

Far Field Approximations

In many practical situation the observation point is typically quite far away from the source region. Simplified results can be constructed.



$$R = |\vec{R}| = r - r'\cos\alpha = r - r'\frac{\vec{r}'}{r'}\cdot\frac{\vec{r}}{r} = r - \vec{r}'\cdot\hat{r}$$

then: $$\frac{e^{-j\beta R}}{4\pi R} = \frac{e^{-j\beta(r-\vec{r}'\cdot\hat{r})}}{4\pi(\underbrace{r-\vec{r}'\cdot\hat{r}}_{\text{much smaller than } r})} \approx \frac{e^{-j\beta r}}{4\pi r}\, e^{j\beta\vec{r}'\cdot\hat{r}}$$

Far field approximation: $$\boxed{\vec{A} \simeq \mu \frac{e^{-j\beta r}}{4\pi r}\int \vec{J}\, e^{j\beta\hat{r}\cdot\vec{r}'}\, dv'}$$ Very important result

$\vec{E}$ & $\vec{H}$ fields: $\vec{E} \simeq -j\omega(A_\theta\hat{\theta} + A_\phi\hat{\phi})$; $\vec{H} = \frac{1}{\eta}\hat{r}\times\vec{E}$

Uniform Plane Waves
Oblique angles (1)

Maxwell's eqs., lossless & source free

$$\nabla \times \vec{E} = -j\omega \vec{B} \qquad \nabla \cdot \vec{D} = 0$$

$$\nabla \times \vec{H} = j\omega \vec{D} \qquad \nabla \cdot \vec{B} = 0$$

Propagation vector — some books use $\vec{k}$

Assume plane wave behavior $\Rightarrow$ all field $\propto e^{-j\vec{\beta}\cdot\vec{r}}$

Position vector

Note: $\vec{\beta}\cdot\vec{r} = (\beta_x \hat{x} + \beta_y \hat{y} + \beta_z \hat{z}) \cdot (x\hat{x} + y\hat{y} + z\hat{z})$

Performing spatial differentiation, one finds:

$$\boxed{\nabla \longrightarrow -j\vec{\beta}}$$

An important result applicable to plane waves

Uniform Plane Waves
oblique angles (2)

Now Maxwell's Eqs. becomes:

$$-j\vec{\beta}\times\vec{E} = -j\omega\vec{B} \qquad\qquad \vec{\beta}\times\vec{E} = \omega\vec{B}$$

$$-j\vec{\beta}\times\vec{H} = j\omega\vec{D} \qquad\qquad -\vec{\beta}\times\vec{H} = \omega\vec{D}$$

$$-j\vec{\beta}\times\vec{D} = 0 \qquad\qquad \vec{\beta}\cdot\vec{D} = 0 \qquad \vec{\beta}\perp\vec{D}$$

$$-j\vec{\beta}\cdot\vec{B} = 0 \qquad\qquad \vec{\beta}\cdot\vec{B} = 0 \qquad \vec{\beta}\perp\vec{B}$$

orthogonal

Important observation: Even in complex media where $\vec{D}\neq\epsilon\vec{E}$, and $\vec{B}\neq\mu\vec{H}$ both $\vec{D}$ and $\vec{B}$ are orthogonal to $\vec{\beta}$.

This leads to the KDB, (K used for β) system for plane wave analysis in complex media.

Uniform Plane Waves
oblique angles (3)

Let us consider simple media for now

$$\vec{D} = \epsilon \vec{E} \; ; \; \vec{B} = \mu \vec{H}$$

scalar

Then: $\vec{\beta} \times \vec{E} = \omega\mu \vec{H}$

$$-\vec{\beta} \times \vec{H} = \omega\epsilon \vec{E}$$

$$\vec{\beta} \cdot \vec{E} = 0$$

$$\vec{\beta} \cdot \vec{H} = 0$$

$$\Rightarrow \quad \vec{\beta} \perp \vec{E} \qquad \vec{\beta} \perp \vec{H}$$

How about $\vec{E}$ and $\vec{H}$

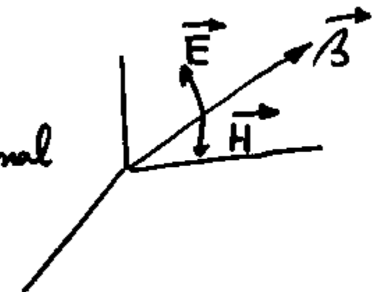
Note: $\omega\mu \vec{H} \cdot \vec{E} = (\vec{\beta} \times \vec{E}) \cdot \vec{E}$

$$= \vec{\beta} \cdot (\vec{E} \times \vec{E}) = 0$$

Then $\vec{E} \perp \vec{H}$

$\vec{E}$ is prependicular to $\vec{H}$.

Uniform Plane Waves
oblique angles (4)



Important result

$(\vec{\beta}, \vec{E}, \vec{H})$ create orthogonal vectors.

$\vec{\beta}\cdot\vec{E}=0$

$\vec{\beta}\cdot\vec{H}=0$ and $\boxed{\vec{H}=\frac{1}{\omega\mu}\vec{\beta}\times\vec{E}}$

$\vec{E}\cdot\vec{H}=0$

For plane waves, the wave-eq. becomes:

$$\vec{\beta}\times(\vec{\beta}\times\vec{E}) = \omega\mu(\vec{\beta}\times\vec{H}) = \omega\mu(-\omega\epsilon\vec{E})$$

$$\underbrace{(\vec{\beta}\cdot\vec{E})}_{=0}\vec{\beta} - (\vec{\beta}\cdot\vec{\beta})\vec{E} = -\omega^2\mu\epsilon\vec{E}$$

$$\Rightarrow (\vec{\beta}\cdot\vec{\beta})\vec{E} = \omega^2\mu\epsilon\vec{E}$$

$$\vec{\beta}\cdot\vec{\beta} = \omega^2\mu\epsilon$$

or $\boxed{\beta_x^2+\beta_y^2+\beta_z^2=\omega^2\mu\epsilon=k^2}$

Observation: β_z is not independent of $\beta_x, \beta_y \Rightarrow \boxed{\beta_z^{\pm}=\pm\sqrt{\omega^2\mu\epsilon-\beta_x^2-\beta_y^2}}$

$\pm$ sign for waves propagating in $\pm z$ direction

Polarization (1)

For a plane wave propagating in $\hat{z}$ direction

phase

$$\text{Phasor:}\quad \vec{E} = (\hat{x}E_x + \hat{y}E_y)\, e^{-j\beta_z z} = (\hat{x}|E_x|\, e^{j\varphi_x} + \hat{y}|E_y|\, e^{j\varphi_y})\, e^{-j\beta z}$$

amplitude

$$\text{Instantaneous field:}\quad \vec{\mathcal{E}} = \mathrm{Re}[\vec{E}\, e^{j\omega t}] = \hat{x}|E_x|\cos(\omega t - \beta z + \varphi_x) + \hat{y}|E_y|\cos(\omega t - \beta z + \varphi_y)$$

Depending on the values of φ_x and φ_y, the tip of vector $\vec{\mathcal{E}}$ may trace many different paths.

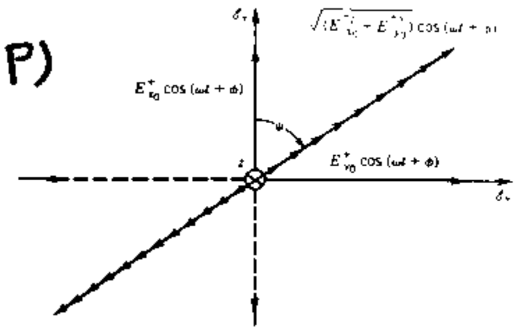
Polarization designates the state of this vector

Polarization (2)

State 1: Linear polarization (LP)

Let $\varphi_x = \varphi_y = \varphi$

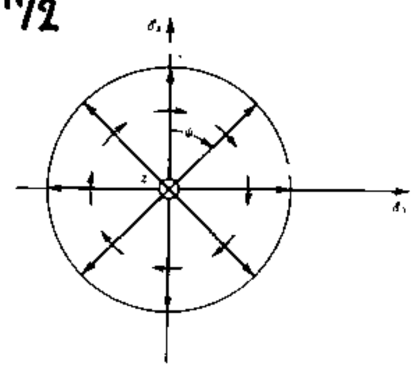
$$\psi = \tan^{-1} \frac{|E_y|}{|E_x|}$$



State 2: Right-Hand Circular polarization (RHCP)

Let $\varphi_y = \varphi_x - \pi/2$

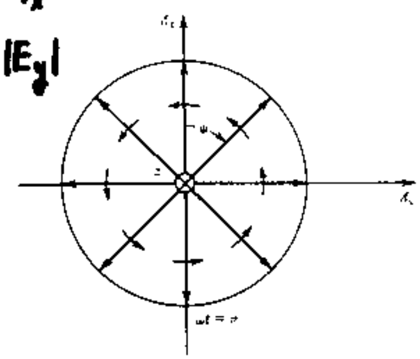
$|E_x| = |E_y|$



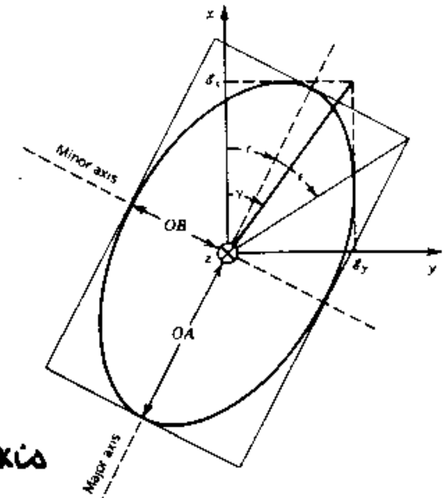
State 3: Left-Hand Circular polarization (LHCP)

Let $\varphi_y = \varphi_x + \pi/2$

$|E_x| = |E_y|$



Polarization (3)



State 4: Elliptical polarization

Let φ_y, φ_x arbitrary

$|E_x| > |E_y|$ arbitrary

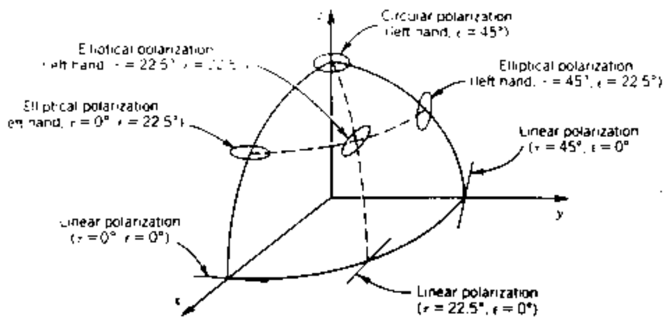
For this arbitrary wave define

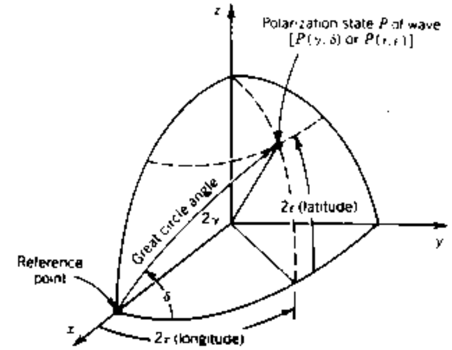
$$AR = \text{Axial Ratio} = \pm \frac{\overbrace{OA}^{\text{major axis}}}{\underbrace{OB}_{\text{minor axis}}} \Longrightarrow 1 \le |AR| \le \infty$$

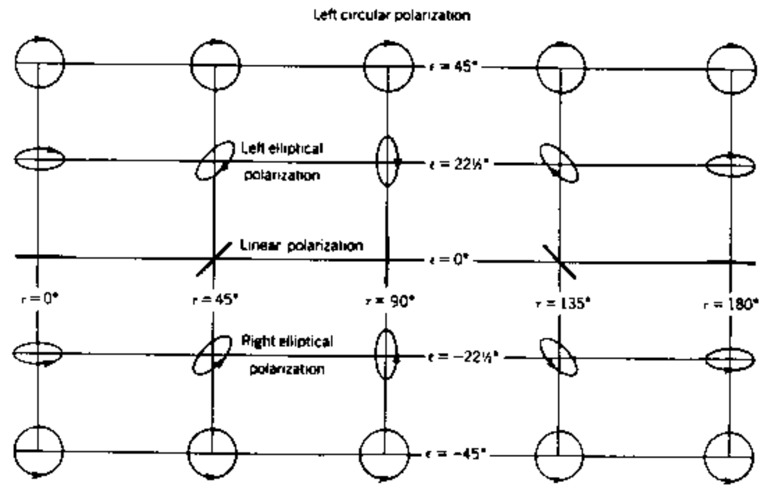
$\pm$ ⟹ + for RH polarization
− for LH polarization

$$\underbrace{\tau}_{\text{tilt angle}} = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{2|E_x||E_y|}{|E_x|^2 - |E_y|^2}\cos\Delta\varphi\right) \Longrightarrow \Delta\varphi = \varphi_x - \varphi_y$$

Poicare Sphere (2)







Polarization (4)
Poincare' Sphere (1)

Poincare' sphere is a novel demonstration mechanism to show various polarization states of waves

Define (γ, δ) set

$$\gamma = \tan^{-1}\frac{|E_y|}{|E_x|} \text{ or } \tan^{-1}\frac{|E_x|}{|E_y|}$$

$$0^\circ \le \gamma \le 90^\circ$$

$$\delta = \varphi_y - \varphi_x$$

$$-180^\circ \le \delta \le 180^\circ$$

2γ = great circle angle

Define (ε, τ) set

$$\varepsilon = \cot^{-1}(AR)$$

$$-45^\circ \le \epsilon \le 45^\circ$$

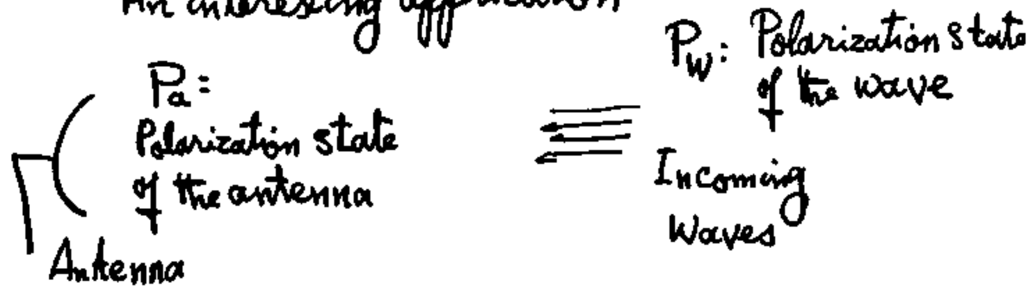
τ = tilt angle (see above)

2ϵ = latitude

2τ = longitude

Poincare Sphere (3)

An interesting application



Voltage response of the antenna : $V = C \cos \frac{\angle P_W P_a}{2}$

C = Constant depending on antennas size, etc.

$\angle P_W P_a$ = Angle subtended by a great-circle arc from polarization P_W to P_a

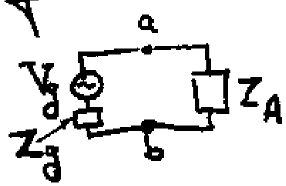
See example 4-12 & 4-13 in the book.

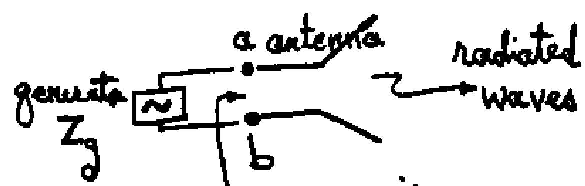
Input Impedance of Antennas

- We need signal source generator to sustain the oscillation of the current on the antenna.

- Input impedance is defined: The impedance presented by an antenna at its terminals.

Equivalent Circuits



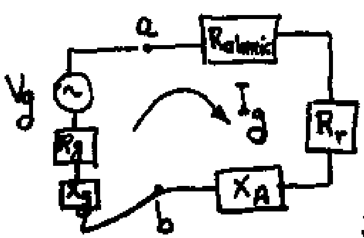


$$Z_A = R_A + jX_A$$

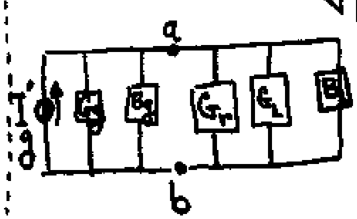
$R_r + R_{ohmic}$

radiation resistance

Thevenin



Norton



Input Impedance of Antennas

$$I_g = \frac{V_g}{Z_A + Z_g} = \frac{V_g}{(R_r + R_{ohmic} + R_g) + j(X_A + X_g)}$$ Amp.

$|V_g|$ is the peak voltage

then:

$$P_r = \frac{1}{2}|I_g|^2 R_r = \frac{|V_g|^2}{2}\left[\frac{R_r}{(R_r + R_{ohmic} + R_g)^2 + (X_A + X_g)^2}\right]$$

Powered delivered to the antenna for radiation $\frac{1}{2}\int Re(\vec{E} \times \vec{H}^*) \cdot d\vec{s}$

in W

$$P_L = \frac{1}{2}|I_g|^2 R_{ohmic} = \frac{|V_g|^2}{2} \frac{R_{ohmic}}{\text{same as above}}$$

Power dissipated as heat

$$P_g = \frac{1}{2}|I_g|^2 R_g = \frac{|V_g|^2}{2} \frac{R_g}{\text{same as above}}$$

Power dissipated as heat on the interal resistance of the generator

As you know from circuit courses: the maximum power delivered to the antenn when there is a conjugate match :

$Z_A = Z_g^*$ ⟹ Conjugate

or $\boxed{\begin{aligned} R_r + R_{ohmic} &= R_g \\ X_A &= -X_g \end{aligned}}$

Input Impedance of Antennas

Under conjugate match situation:

$$P_r = \frac{|V_g|^2}{8} \frac{R_r}{(R_r + R_{ohmic})^2}, \quad W$$

$$P_L = \frac{|V_g|^2}{8} \frac{R_{ohmic}}{(R_r + R_{ohmic})^2}, \quad W$$

$$P_g = \frac{|V_g|^2}{8} \frac{R_g}{(R_r + R_{ohmic})^2} = \frac{|V_g|^2}{8 R_g}, \quad W$$

Recall $R_g = R_r + R_L$

It is clear that $\boxed{P_g = P_r + P_L}$

Power supplied by the generator: $P_s = \frac{1}{2} V_g I_g^* = \frac{|V_g|^2}{4} \frac{1}{R_r + R_L}, \quad W$

Observation: Under conjugate match condition, of the power that is supplied by the generator, half is dissipated as heat in the internal resistance R_g and the other half is delivered to the antenna.

Antenna Radiation Efficiency

$$e_r \,(\text{or } \eta_r) \doteq \frac{\text{Power radiated by the antenna}}{\text{Power delivered to the antenna}} = \frac{P_r}{P_r + P_{ohmic}}$$

then

$$e_r = \frac{\frac{1}{2} R_r |I_g|^2}{\frac{1}{2} R_r |I_g|^2 + \frac{1}{2} R_{ohmic} |I_g|^2} = \frac{R_r}{R_r + R_{ohmic}} = \frac{R_r}{R_A}$$

For many antennas, this radiation efficiency is nearly 100%.

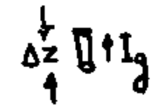
For small antennas in terms of the wavelength, this radiation efficiency can be low. This means that the antenna is not an effective radiator of EM waves.

Radiation Resistance of an Ideal Dipole

Recall: $P_r = \frac{1}{2}|I_g|^2 R_r \Rightarrow R_r = \frac{2}{|I_g|^2} \underbrace{P_r}_{\text{radiated Power}}$

Recall: $P_r = \frac{1}{2} \mathrm{Re} \int \vec{E} \times \vec{H}^* \cdot d\vec{s}$

Also recall: $P_r = \frac{1}{2\eta} \int_0^{2\pi} \int_0^{\pi} (|E_\theta|^2 + |E_\phi|^2)\, \underbrace{r^2 \sin\theta\, d\theta\, d\phi}_{d\Omega}$



① Write the far field: $\vec{E} = j\omega\mu I_g \Delta z \frac{e^{-j\beta r}}{4\pi r} \sin\theta\, \hat{\theta}$

② $P_r = \frac{1}{2\eta} (\omega\mu I_g \Delta z)^2 \cdot \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{\pi} \sin^3\theta\, d\theta\, d\phi = \frac{\omega\mu\beta}{12\pi} (I_g \Delta z)^2$

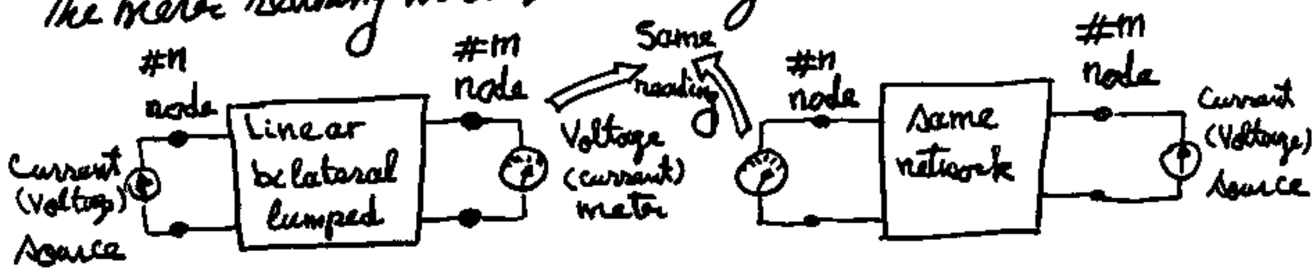
Finally: $\boxed{R_r = \frac{2}{|I_g|^2} P_r = 80\pi^2 \left(\frac{\Delta z}{\lambda}\right)^2}$

Since $\Delta z \ll \lambda$ for an ideal dipole, R_r is very small.

Note: $\eta = \sqrt{\frac{\mu}{\epsilon}}$, $\beta = \omega\sqrt{\mu\epsilon}$, $\beta = 2\pi/\lambda$

Statement of Reciprocity in Circuits

" In any network composed of linear, bilateral, lumped elements, if one places a constant <u>Current</u> (voltage) generator between <u>two nodes</u> (in any branch) and places a <u>Voltage</u> (current) meter between any other <u>two nodes</u> (in any other branch), makes observation of the meter reading, then interchanging the locations of the source and the meter, the meter reading will be unchanged".



Statement of Reciprocity for Antennas (1)

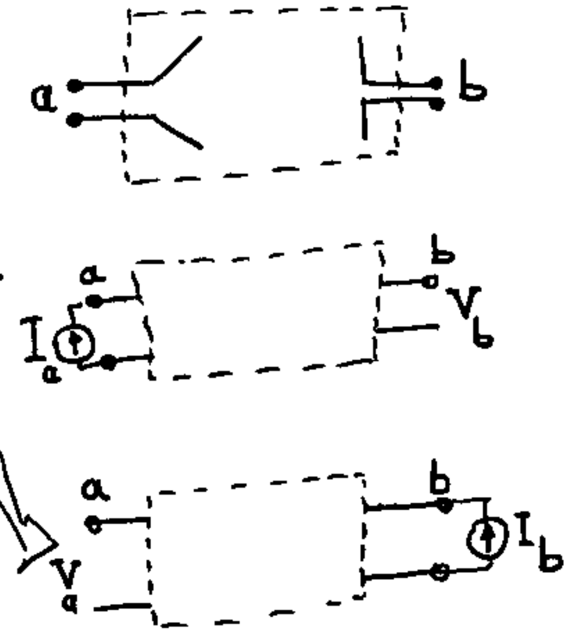
The reciprocity theorem in a circuit is well known. Its application to antennas, however, is not simple. This is due to the fact that outside the antenna, either the radiated field of the transmitting antenna or the incident field on the receiving antenna are vector fields characterized by polarization and spatial variation, which are not describable by circuit quantities. Hence the reciprocity for an antenna <u>cannot</u> be simply stated by exchange of sources and meters.

Statement of Reciprocity for Antennas (2)

(a) Transfer (or mutual) impedance between two antennas

$$Z_{ba} = \frac{V_b}{I_a}\bigg|_{I_b=0}$$

$$Z_{ab} = \frac{V_a}{I_b}\bigg|_{I_a=0}$$



For many practical cases,

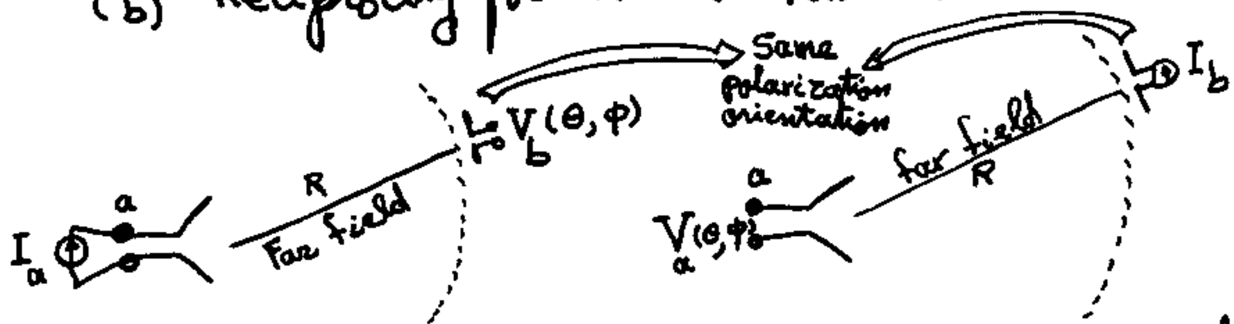
Lorentz reciprocity theorem

$$\iiint_{v_a} \vec{E}_b \cdot \vec{J}_a \, dv' = \iiint_{v_b} \vec{E}_a \cdot \vec{J}_b \, dv'$$

results : $\boxed{Z_{ab} = Z_{ba} = Z_m}$ → defined as mutual impedance

Statement of Reciprocity for Antennas (3)

(b) Reciprocity for radiation Pattern



Transmitting pattern of antenna "a"   Receiving pattern of antenna "a"

$$Z_{ab}(\theta,\phi) = \frac{V_b(\theta,\phi)}{I_a} \qquad Z_{ba}(\theta,\phi) = \frac{V_a(\theta,\phi)}{I_b}$$

Then :

$$\boxed{Z_{ab}(\theta,\phi) = Z_{ba}(\theta,\phi)}$$

Observation: The transmit and receive patterns of an antenna are identical.

Extremely important result

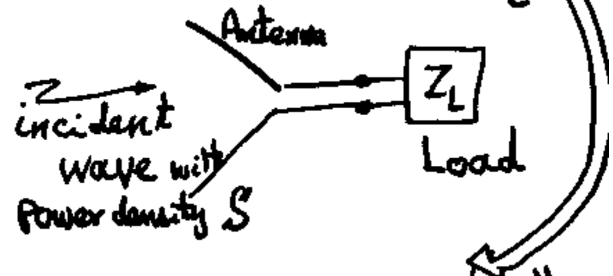
Antennas in Receiving Mode (1)

So far, we have examined antennas as "radiator" of the energy supplied by a source.

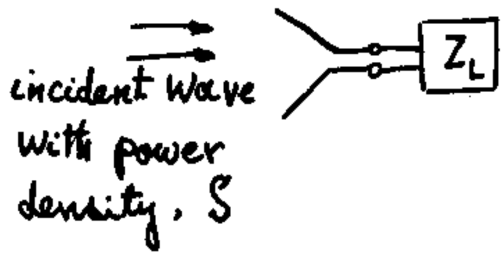
Question: How a "receiving" antenna extracts energy from an incident wave and delivers it to a load?

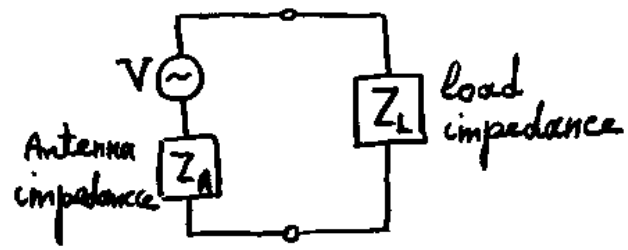
$$P = S A_e$$

The ability of an antenna to capture energy from an incident wave and convert it into an intercepted power for delivery to a load is characterized by "effective aperture"



Antennas in Receiving Mode (2)

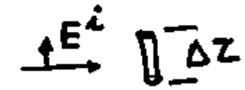




The power available from the antenna is realized when the antenna impedance is matched by a load impedance of $Z_L = R_r - jX_A$ if we assume $R_{ohmic} = 0$. Then

$$\underbrace{P_{Am}}_{\text{maximum available power}} = \frac{1}{8}\frac{|V_A|^2}{R_r} = \frac{1}{8}\frac{|E^i|^2}{R_r}(\Delta Z)^2$$

Example:
Ideal Dipole



$V_A = E^i \Delta Z$

for ideal dipole

$R_r = \eta \frac{2}{3}\pi\left(\frac{\Delta Z}{\lambda}\right)^2$

Antennas in Receiving Mode (3)

Power density ~~of~~ in the incoming wave

$$\mathcal{S} = \frac{1}{2}|\vec{E} \times \vec{H}^*| = \frac{1}{2}\frac{|E^i|^2}{\eta}$$

Remember, $\vec{E}$ & $\vec{H}$ have local plane wave charateristics

η ⇒ Free space impedance

Let us define : $P_{Am} = \mathcal{S} A_{em}$ ⇒ maximum effective apertune

then : $A_{em} = \dfrac{P_{Am}}{\mathcal{S}} = \left(\dfrac{3}{8\pi}\right)\lambda^2$: for ideal dipole ($\dfrac{3}{8\pi} \to 0.119$)

Recall : $D = \frac{3}{2}$: directivity of the ideal dipole

Finally : $\boxed{\mathcal{D} = \dfrac{4\pi}{\lambda^2} A_{em}}$ Extremely important result Applies to any antenna

Antennas in Receiving Mode (4)

radiation efficiency

Effective aperture : $A_e = e_r A_{em}$ maximum effective aperture

gain

Recall : $G = e_r D$ directivity

Very important

Recall : $D = \frac{4\pi}{\lambda^2} A_{em} \implies \boxed{G = \frac{4\pi}{\lambda^2} A_e}$

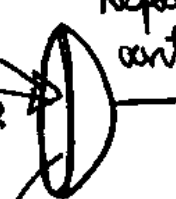
Physical aperture

For electrically large antennas : $A_e = \varepsilon_{ap} A_p$

aperture efficiency

usully 50% to 70%

πa^2



Communication Links: Friis Formula

$$P_r = S \, A_{er}$$

S: Power density by satellite

A_{er}: effective aperture of received antenna

$$S = \frac{P_t}{4\pi R^2} G_t$$

S: W/m^2

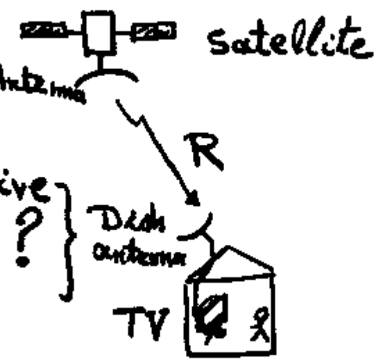
P_t: total transmitted power

G_t: gain of transmitting antenna

R: distance between transmitting & receiving antennas

Transmit P_t, G_t } Antenna

Receive P_r ?, G_r } Dish antenna

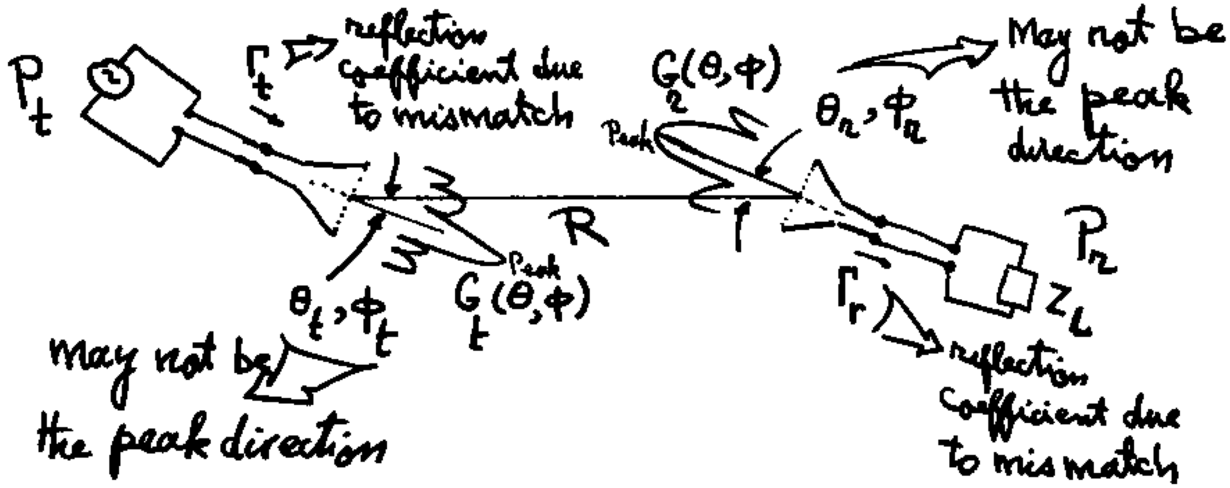


Question: What is the received power under the matched conditions?

Recall: $G_r = \frac{4\pi}{\lambda^2} A_{er} \Rightarrow \boxed{A_{er} = \frac{\lambda^2}{4\pi} G_r}$

Friis transmission formula: $\boxed{P_r = P_t \left(\frac{\lambda}{4\pi R}\right)^2 G_t G_r}$ or $\boxed{P_r = P_t \frac{1}{(R\lambda)^2} A_{et} A_{er}}$

Communication Links: Generalization



$$\boxed{\frac{P_r}{P_t} = (1-|\Gamma_t|^2)(1-|\Gamma_r|^2)\left(\frac{\lambda}{4\pi R}\right)^2 G_t(\theta_t,\phi_t)\,G_r(\theta_r,\phi_r)\cdot\underbrace{|\hat{\rho}_t\cdot\hat{\rho}_r|^2}_{\text{due to polarization mismatch}}}$$

This is the basis for the most of Communication system designs.

Communication Link in dBm unit

Definition: dBm is power in decibels above milliwatt.

example: $1^W \equiv 30\,dBm$ $\left(= 10 \log \overset{\text{milliwat}}{1000}\right)$ $1W$

Free space loss factor

Friis eq.: $P_r = P_t \left(\frac{\lambda}{4\pi R}\right)^2 G_t G_r$

$G(dB) = 10 \log G$

$$P_r(dB_m) = P_t(dB_m) + G_t(dB) + G_r(dB) - 20\log(R)$$
$$- 20 \log f(MHz) - 32.44$$

Note: $\lambda = \frac{c}{f}$ — c: velocity of light; f: Freq.

Note: Same formula also applies for P_r (dBW)

EIRP: Effective Isotropically Radiated Power

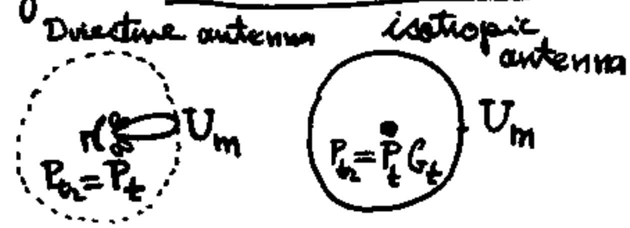
EIRP is a parameter used in the broadcast industry. FM radio stations often mention their effective radiated power when they sign off at night.

Definition: $\boxed{EIRP = \underbrace{P_t}_{\text{transmitted power}} \; \underbrace{G_t}_{\text{antenna maximum gain}}}$

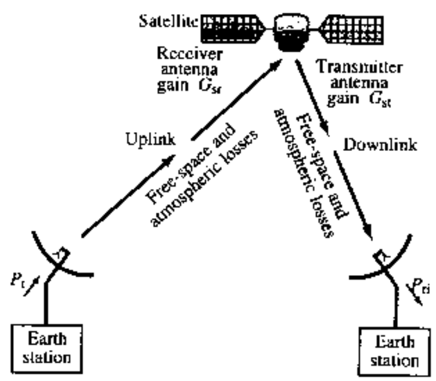
Recall: $G_t = \dfrac{4\pi \overbrace{U_m}^{\text{radiation intensity}}}{P_t} \Rightarrow \boxed{EIRP = 4\pi U_m}$

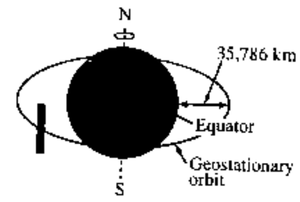
What does it mean?

To obtain the same radiation intensity produced by directional antenna, an isotropic antenna would have an input power G_t times greater.

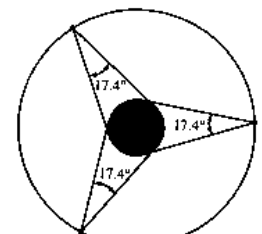


Geostationary Satellite Communications

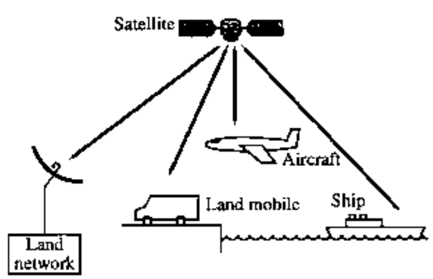


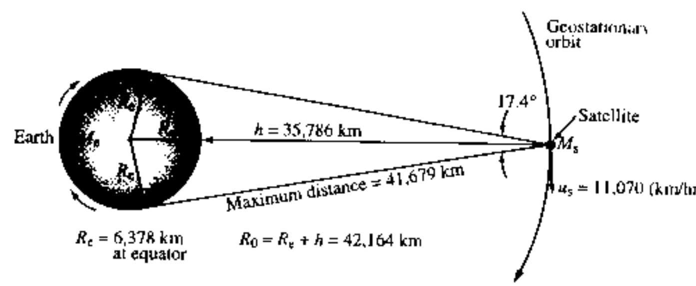


Geostationary satellite orbit

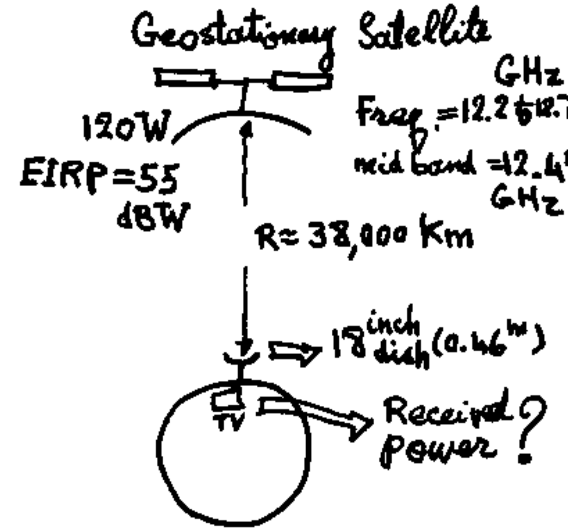


Worldwide coverage by three satellites spaced 120° apart





Example : DBS -- Direct Broadcast Satellite @Ku band



Find the received power?

- $f = 12.45\ GHz$ (mid freq.)
- $P_r(dBW) = 20.8\ dBW$ $(10\log 120)$
- $G_t(dB) = EIRP(dBW) - P_t(dBW)$
  $= 55 - 20.8 = 34.2\ dB$
- $R = 38{,}000\ Km$
- $G_r = \frac{4\pi}{\lambda^2}\varepsilon_{ap}A_p = \frac{4\pi}{(0.024)^2}\,0.7\,\left(\pi\left(\frac{0.46}{4}\right)^2\right)$ 70% eff. area of the 18" inch diameter dish
  $= 2538 \Rightarrow G_r(dB) = 34\ dB$
- $P_r(dBW) =$ [Friis eq. in dBW) $\Rightarrow$ $\left\{ \begin{array}{l} 20.8 + 34.2 + 34 - 91.6 - 81.9 - 32.4 \\ = -116.9\ dBW \Rightarrow \boxed{2\times10^{-12}\ W} \end{array} \right.$

Without total antenna gain of $G_t + G_t = 68dB$ this signal is hopelessly lost in noise.

Example: DBS -- Signal to noise ratio

Due to electromagnetic noise picked up by the antenna as well as noise generated by the receiver electronics, a home receiving TV antenna system has a noise level given by

Boltzmann's constant (1.38×10^{-23} J/K)

$$\boxed{P_n = k\, T_{sys}\, B \ (W)}$$

System noise temperature in kelvin (K)

B is the receiver bandwidth in Hz

For ground DBS antenna: $T_{sys} = 125\,K$

$B = 20$ MHz

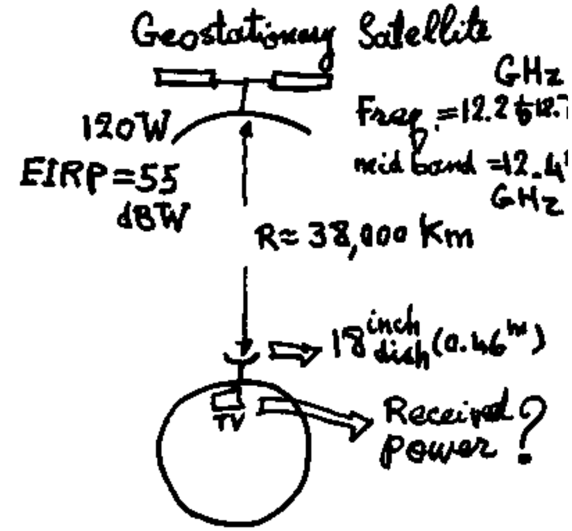
then: $P_n = 1.38 \times 10^{-23} \cdot 125 \cdot 20 \times 10^{6} = 3.5 \times 10^{-14} = -134.6$ dBW

Signal to noise ratio or Carrier to noise ratio: $S_n = \dfrac{P_r}{P_n}$ ⇒

$$\boxed{\begin{aligned} S_n(dB) &= CNR(dB) = P_r(dBW) - P_n(dBW) \\ &= -116.9 - (-134.6) = 17.7\,dB \end{aligned}}$$

Reasonable margin

Example : DBS -- Direct Broadcast Satellite @Ku band



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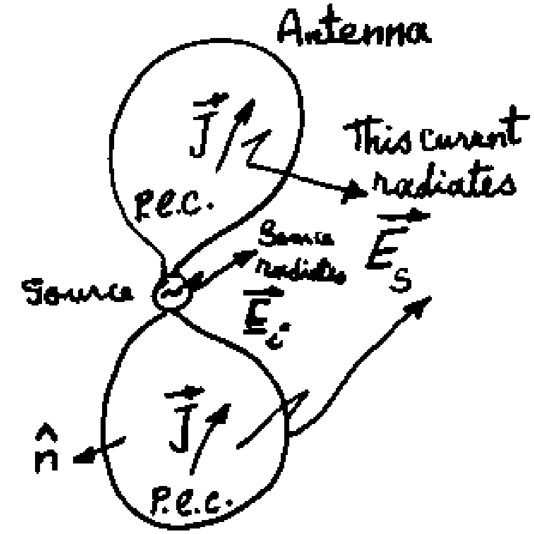
Reasonable margin

Computational Electromagnetics (CEM): Method of Moments (1)

For simple structures, one may be able to guess the functional behavior the current density. However, for not very simple structures, one needs to determine the proper form of the current. This, usually, necessitates to apply numerical methods for the solution of the Maxwell's eqs.

Among various numerical (computational) techniques, the Method of Moments (or MoM) has received tremendous attention in solving antenna problems.

Method of Moments (2)



The rf source establishes current $\vec{J}$ on the antenna's metalic surface.

The rf source radiates the so called "incident" field $\vec{E}_i$ in the absense of antenna.

The current $\vec{J}$ induced on the antenna radiates the so called "scattered" field $\vec{E}_S$.

In order for the boundary condition on the perfectly electrically conducting (P.E.C.) antenna surface is satisfied, one requires that

$$\hat{n} \times (\vec{E}^i + \vec{E}^s) = 0$$

on the P.E.C. surface of antenna

unit normal to the surface

this gives total tangential E field on the surface.

Method of Moments (3)

Satisfaction of the boundary condition: $\hat{n} \times (\vec{E}^i + \vec{E}^s) = 0$

$\hat{n}$ — known quantity

$\vec{E}^i$ — known quantity

$\vec{E}^s$ — depends on current $\vec{J}$ — Unknown quantity yet to be determined.

Observation: $\vec{E}^s$ is related to $\vec{J}$ via an <u>integral</u> operator.

Therefore, $\hat{n} \times (\vec{E}^i + \vec{E}^s) = 0$ establishes an <u>integral equation</u>.

Example: For a linear antenna

$$\boxed{-\int_{\text{over antenna}} I(z') K(z, z') \, dz' = E^i(z)}$$

$I(z')$ — this current is the unknown.

$E^i(z)$ — over antenna

Method of Moments is a computational technique which allows one to solve integral equations and determine the unknown current.

Method of Moments (4)

Integral equation for a linear antenna:

$$-\int_{\text{over antenna}} \underbrace{I(z')}_{\text{unknown, yet to be determined?}} \underbrace{K(z,z')}_{\text{known}} \, dz' = \underbrace{E^i(z)}_{\text{known}}$$

over the antenna

This is not an easy problem to be done analytically.

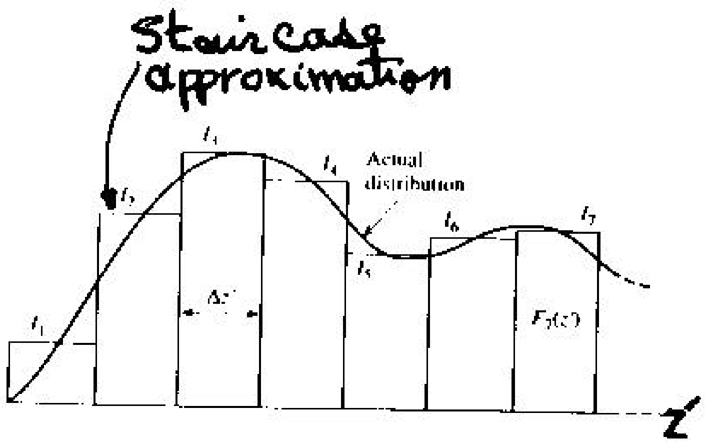
Remember: The objective is to find the functional form of $I(z')$. In other words, what function satisfies the above integral equation.

Method of Moments: MoM procedure allows us to convert the integral eq. into a matrix eq. which can be readily solved.
MoM

Method of Moments (5)

Observation: Sometimes "simple ideas" can be very "powerful".

Idea: Since it is very hard, if not impossible, to get the functional form of $I(z')$, why not breaking it into pieces and then obtain the coefficients of these pieces.



Current distribution on the antenna

$$I(z') = \sum_{n=1}^{N} I_n F_n(z')$$

$F_n(z')$: known function

I_n: unknown coefficients. Yet to be determined.

I_n: does not depend on z'

then:

$$\int_{\text{over antenna}} I(z')K(z,z')\,dz' = \text{this known!}$$

$$\sum_{n=1}^{N} I_n \int_{\text{over antenna}} F_n(z')K(z,z')\,dz'$$

Method of Moments (6)

Substituting in the integral eq. :

$$\underbrace{-\sum_{n=1}^{N} I_n \int F_n(z')K(z,z')\,dz' = E^i(z)}$$

this is valid at every point on the antenna

Note: This is one equation with N unknowns.

In order to create N eqs. and N unknowns, one enforces the equation at N points on the antenna :

Let : $f_n(z) = -\int F_n(z')K(z,z')\,dz'$

F_n: $= 1$ over Δz_n

over the domain of $F_n(z')$

$$I_1 f_1(z_1) + I_2 f_2(z_1) + \cdots + I_N f_N(z_1) = E^i(z_1)$$

$$\vdots$$

$$I_1 f_1(z_m) + I_2 f_2(z_m) + \cdots + I_N f_N(z_m) = E^i(z_m)$$

$$\vdots$$

$$I_1 f_1(z_N) + I_2 f_2(z_N) + \cdots + I_N f_N(z_N) = E^i(z_N)$$

Then : N equations

N unknowns

Method of Moments (7)

In matrix form :

$$\underbrace{\begin{bmatrix} f_1(z_1) & f_2(z_1) & \cdots & f_N(z_1) \\ f_1(z_2) & f_2(z_2) & \cdots & f_N(z_2) \\ \vdots & & & \\ f_1(z_N) & f_2(N) & \cdots & f_N(z_N) \end{bmatrix}}_{\text{known}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}}_{\text{unknown}} = \underbrace{\begin{bmatrix} E^i(z_1) \\ E^i(z_2) \\ \vdots \\ E^i(z_N) \end{bmatrix}}_{\text{known}}$$

Shorthand notation :

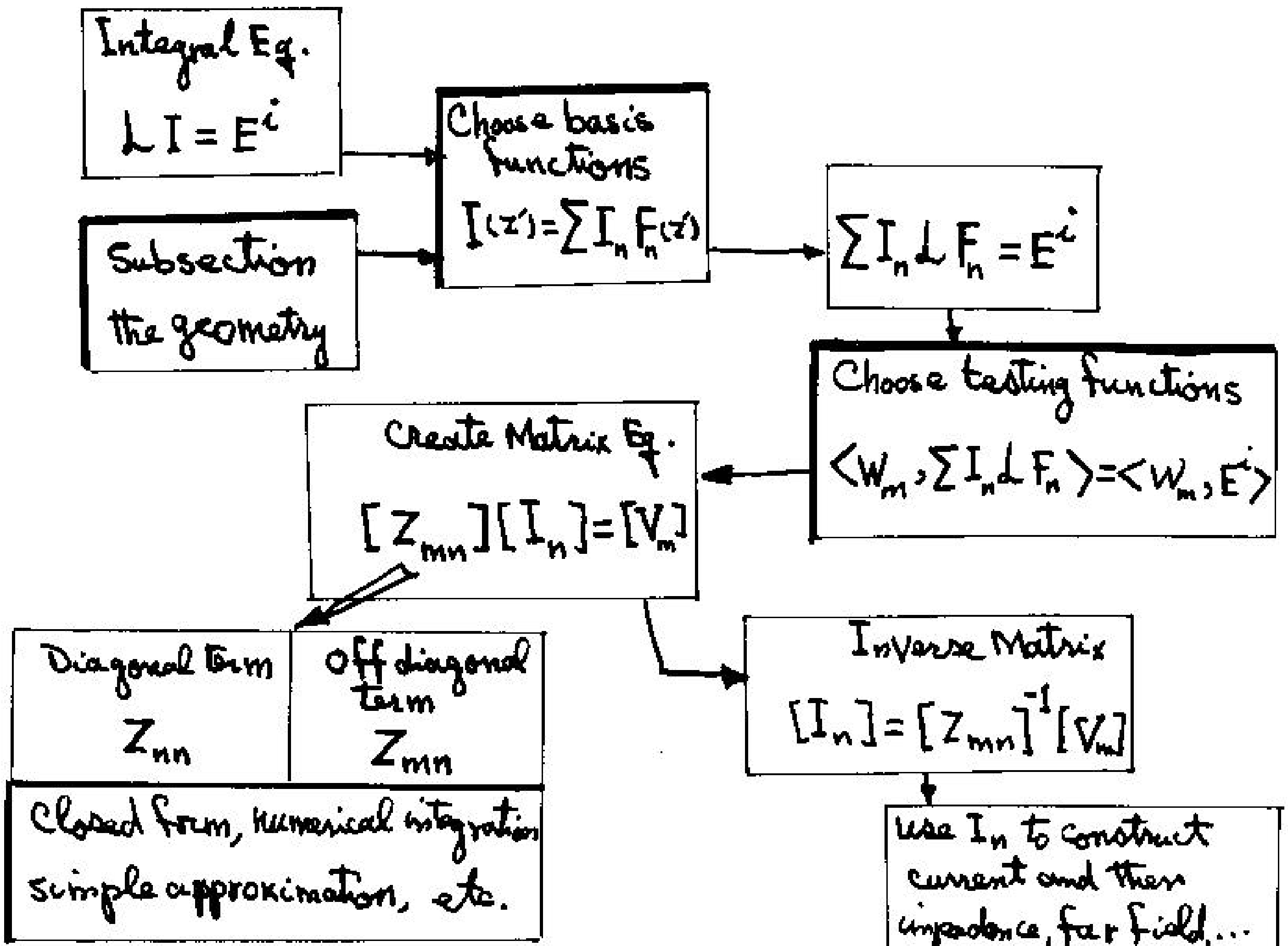
$$\boxed{\underbrace{[Z_{mn}]}_{\substack{\text{square} \\ \text{matrix } N\times N}} \underbrace{[I_n]}_{\text{column matrix } N\times 1} = \underbrace{[V_m]}_{\text{column matrix } N\times 1}}$$

Then :

$$[I_n] = \underbrace{[Z_{mn}]^{-1}}_{\substack{\text{inverse} \\ \text{matrix}}} [V_m]$$

this can be readily solved using any matrix inversion routines.

MoM block diagram (8)



MoM applied to Linear Antennas

Construction of Integral Equation (1)

Recall: $\vec{E} = -j\omega\vec{A} - \nabla\Phi$, $\nabla\cdot\vec{A} + j\omega\mu\epsilon\Phi = 0$, $\vec{A} = \int \mu \vec{J}(\vec{r}')\, g(\vec{r},\vec{r}')\, dv'$

and $g(\vec{r},\vec{r}') = \dfrac{e^{-j\beta|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}$, $|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

Since we will apply the boundary condition along the z direction, let us concentrate on the z components:

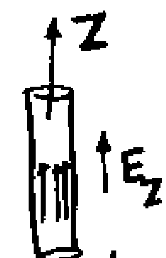
$$E_z = -j\omega A_z - \frac{\partial\Phi}{\partial z}$$

$$\frac{\partial A_z}{\partial z} + j\omega\mu\epsilon\Phi = 0 \Rightarrow \frac{\partial\Phi}{\partial z} = -\frac{1}{j\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial z^2}$$

or

$$\boxed{E_z = \frac{1}{j\omega\epsilon}\left(\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z\right)}$$

$\beta = \omega\sqrt{\mu\epsilon}$



Current is z directed

MoM applied to Linear Antennas
Construction of Integral Eq. (2)

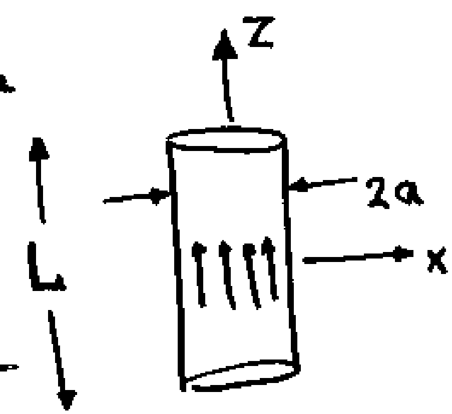
For z oriented current on a cylinder antenna :

$$A_z = \mu \int_0^{2\pi} \int_{-L/2}^{L/2} g(\vec{r}, \vec{r}\,') \, J_z(\vec{r}\,') \, a \, d\phi' dz'$$

For very thin cylinder :

$$J_z(\vec{r}\,') = J_z(\phi', z') = J_z(z')$$

no ϕ' dependence



Let :

$$2\pi a \, J_z(z') = I(z')$$

or :

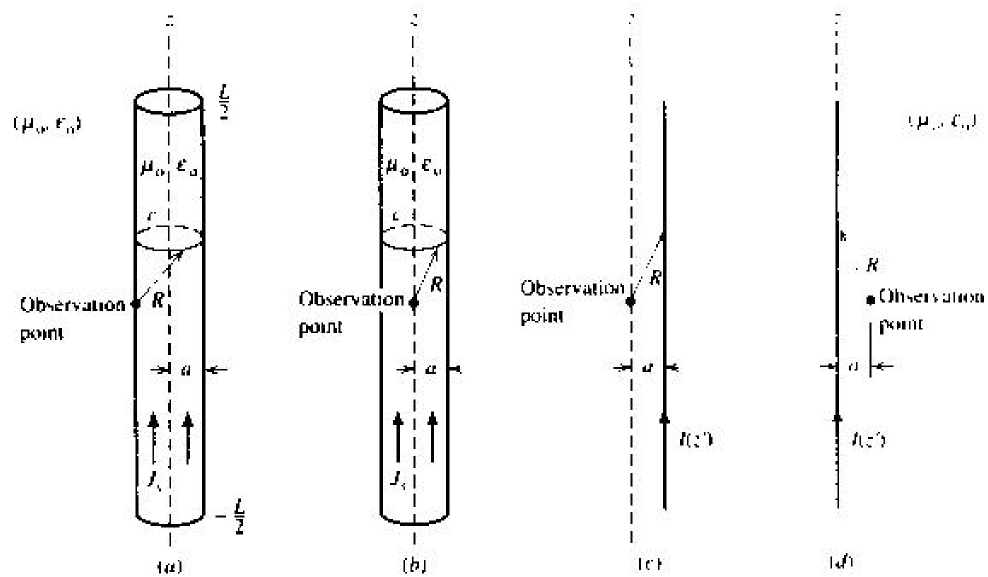
$$\boxed{E_z = \frac{1}{j\omega\epsilon} \int_{-L/2}^{L/2} \left(\frac{d^2 \tilde{g}(z,z')}{dz^2} + \beta^2 \tilde{g}(z,z') \right) I(z') \, dz'}$$

where

$$\tilde{g}(z, z') = \frac{e^{-j\beta R}}{4\pi R} \; ; \quad R = \sqrt{(z-z')^2 + a^2}$$

Observation: Gives E_z based on thin wire approximation.

Thin Wire Antenna Approximation



MoM applied to Linear Antennas

Construction of Integral Eq. (3)

Boundary Condition for P.E.C.: $\hat{n}\times(\vec{E}^i+\vec{E}^s)=0 \Longrightarrow E_z^i+E_z^s=0$ on the thin wire antenna

Finally:

$$\boxed{-\frac{1}{j\omega\epsilon_0}\int_{-L/2}^{L/2}\left[\frac{d^2\tilde{g}(z,z')}{dz^2}+\beta^2\tilde{g}(z,z')\right]I(z')dz'=E_z^i(z)}$$

$I(z')$: Unknown

$E_z^i(z)$: On the thin wire antenna; known

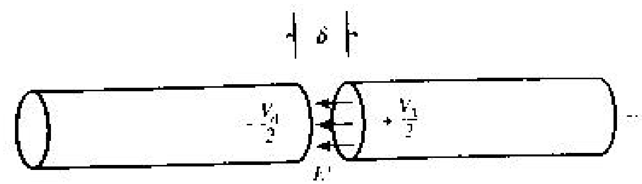
$[\ldots]$: known

This is called Pocklington integral equation for thin wire antenna

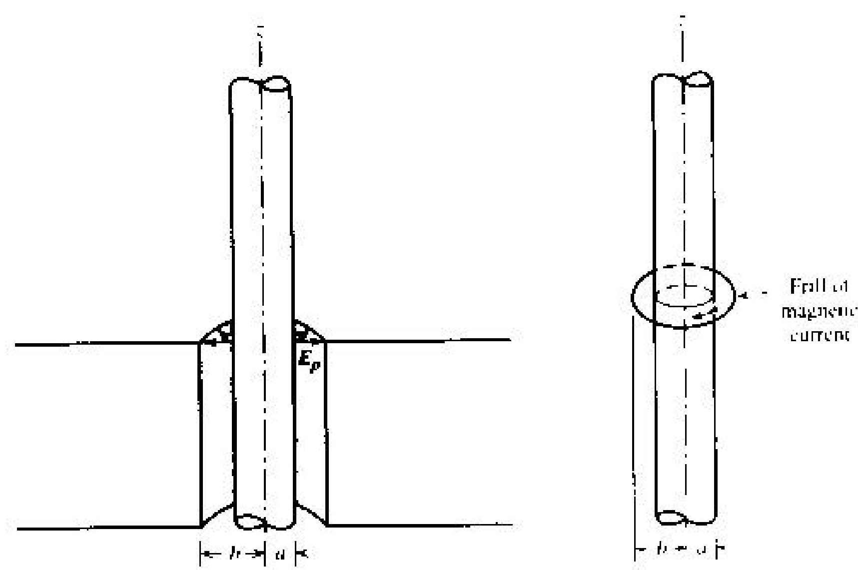
Note: Generalization of this equation can be obtained for any antennas of wire structures or any P.E.C. scattering objects (EE 260B).

Two Popular Feed Models

δ-gap :



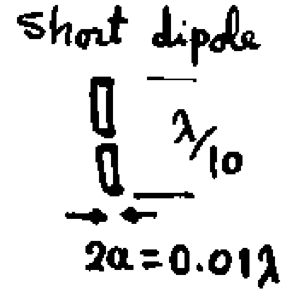
Frill :



(a) Coaxial line feeding a monopole through a ground plane.

(b) Mathematical model

MoM Example: Short dipole

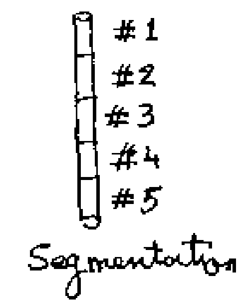


5x5 matrix

$$[Z_{mn}] = 10^{2}\begin{bmatrix} 679.5\ \angle -89.99^\circ & 292.6\ \angle 89.97^\circ & 33.03\ \angle 89.73^\circ & 9.75\ \angle 89.09^\circ & 4.24\ \angle 87.92^\circ \\ 292.6\ \angle 89.97^\circ & 679.5\ \angle -89.99^\circ & 292.6\ \angle 89.97^\circ & 33.03\ \angle 89.73^\circ & 9.75\ \angle 89.09^\circ \\ 33.03\ \angle 89.73^\circ & 292.6\ \angle 89.97^\circ & 679.5\ \angle -89.99^\circ & 292.6\ \angle 89.97^\circ & 33.03\ \angle 89.73^\circ \\ 9.75\ \angle 89.09^\circ & 33.03\ \angle 89.73^\circ & 292.6\ \angle 89.97^\circ & 679.5\ \angle -89.99^\circ & 292.6\ \angle 89.97^\circ \\ 4.24\ \angle 87.92^\circ & 9.75\ \angle 89.09^\circ & 33.03\ \angle 89.73^\circ & 292.6\ \angle 89.97^\circ & 679.5\ \angle -89.99^\circ \end{bmatrix}$$

δ gap feed:

$$[V_m] = \begin{bmatrix} 0 & \angle 0^\circ \\ 0 & \angle 0^\circ \\ 50.0 & \angle 0^\circ \\ 0 & \angle 0^\circ \\ 0 & \angle 0^\circ \end{bmatrix}, \qquad [I_n] = 10^{-3}\begin{bmatrix} 0.52\ \angle 89.54^\circ \\ 0.98\ \angle 89.64^\circ \\ 1.63\ \angle 89.76^\circ \\ 0.98\ \angle 89.64^\circ \\ 0.52\ \angle 89.54^\circ \end{bmatrix}$$



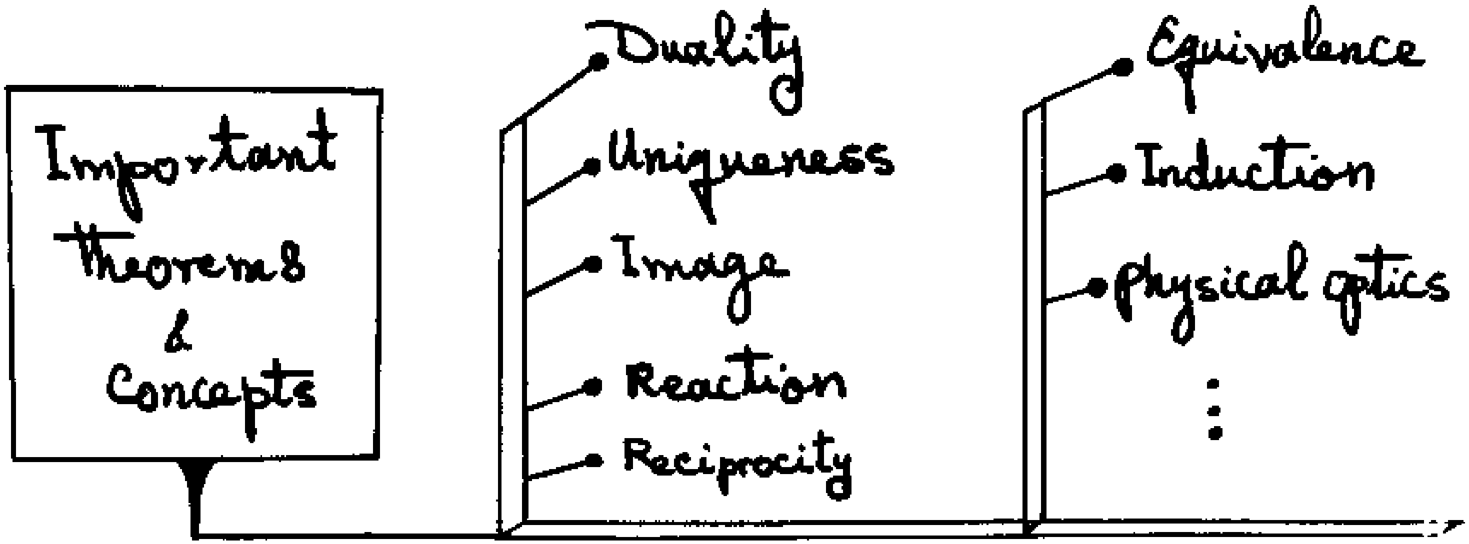
Frill feed:

$$[V_m] = \begin{bmatrix} 0.484\ \angle -0.31^\circ \\ 3.128\ \angle -0.04^\circ \\ 67.938\ \angle -0.002^\circ \\ 3.128\ \angle -0.04^\circ \\ 0.484\ \angle -0.31^\circ \end{bmatrix}, \qquad [I_n] = 10^{-3}\begin{bmatrix} 0.78\ \angle 89.54^\circ \\ 1.48\ \angle 89.64^\circ \\ 2.35\ \angle 89.75^\circ \\ 1.48\ \angle 89.64^\circ \\ 0.78\ \angle 89.54^\circ \end{bmatrix}$$

Current

Theorems & Concepts

There are important Theorems and concepts which facilitate applications and understanding of electromagnetic generation, propagation, radiation, scattering, reception etc.



Duality in E.M.

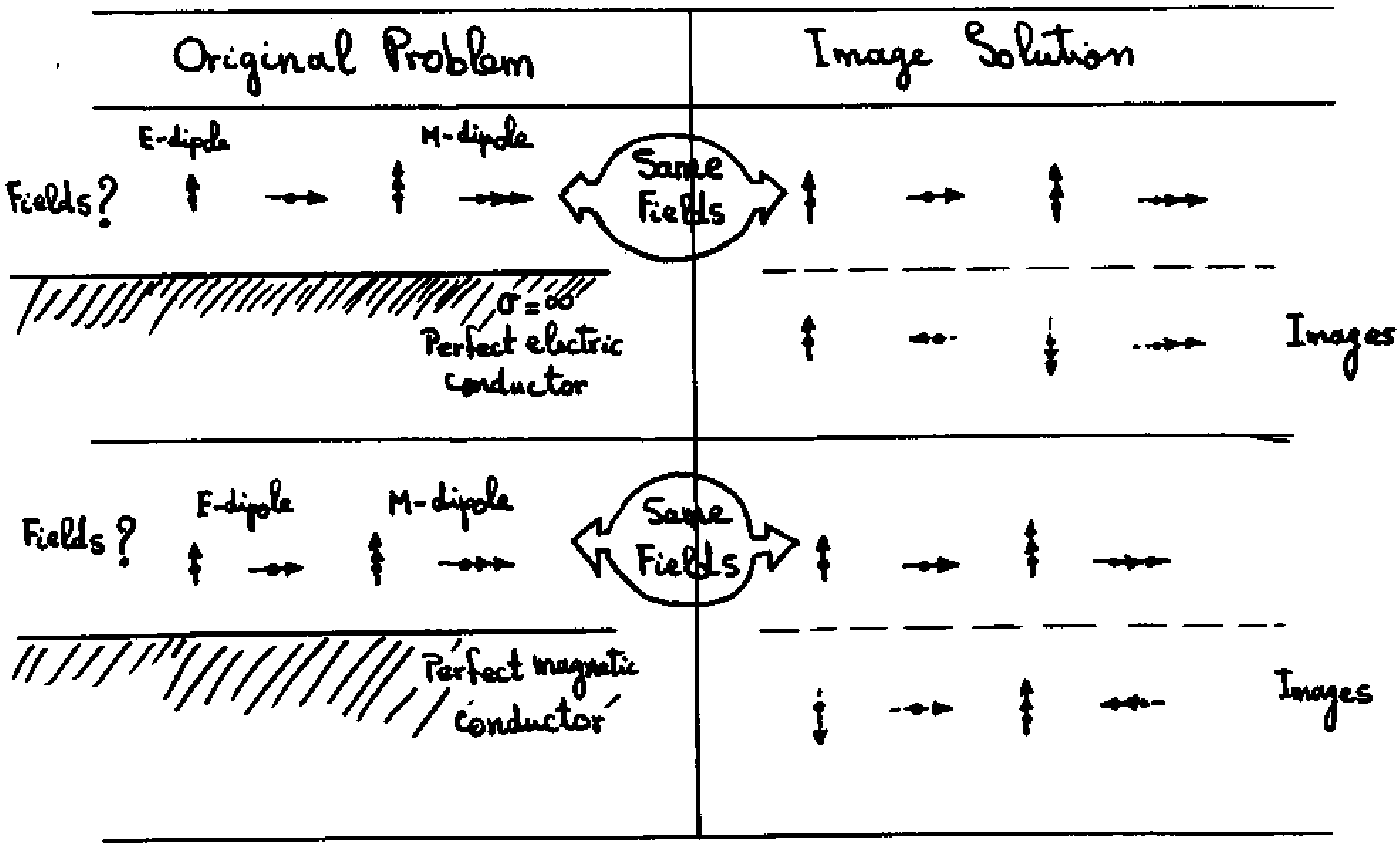
Dual equations for electric (J) and magnetic (M) current sources

Electric sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
$\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A$	$\nabla \times \mathbf{H}_F = j\omega\varepsilon\mathbf{E}_F$
$\nabla \times \mathbf{H}_A = \mathbf{J} + j\omega\varepsilon\mathbf{E}_A$	$-\nabla \times \mathbf{E}_F = \mathbf{M} + j\omega\mu\mathbf{H}_F$
$\nabla^2\mathbf{A} + \beta^2\mathbf{A} = -\mu\mathbf{J}$	$\nabla^2\mathbf{F} + \beta^2\mathbf{F} = -\varepsilon\mathbf{M}$
$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-j\beta R}}{R}\, dv'$	$\mathbf{F} = \frac{\varepsilon}{4\pi}\iiint_V \mathbf{M}\frac{e^{-j\beta R}}{R}\, dv'$
$\mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A}$	$\mathbf{E}_F = -\frac{1}{\varepsilon}\nabla \times \mathbf{F}$
$\mathbf{E}_A = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{A})$	$\mathbf{H}_F = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{F})$

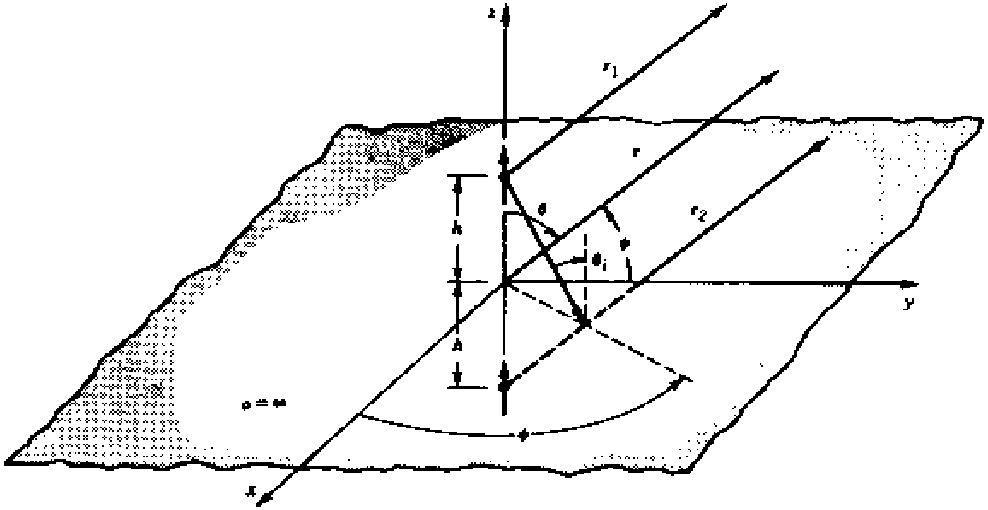
Dual quantities for electric (J) and magnetic (M) current sources

Electric sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
$\mathbf{E}_A$	$\mathbf{H}_F$
$\mathbf{H}_A$	$-\mathbf{E}_F$
$\mathbf{J}$	$\mathbf{M}$
$\mathbf{A}$	$\mathbf{F}$
ε	μ
μ	ε
β	β
η	$1/\eta$
$1/\eta$	η

Image Theory

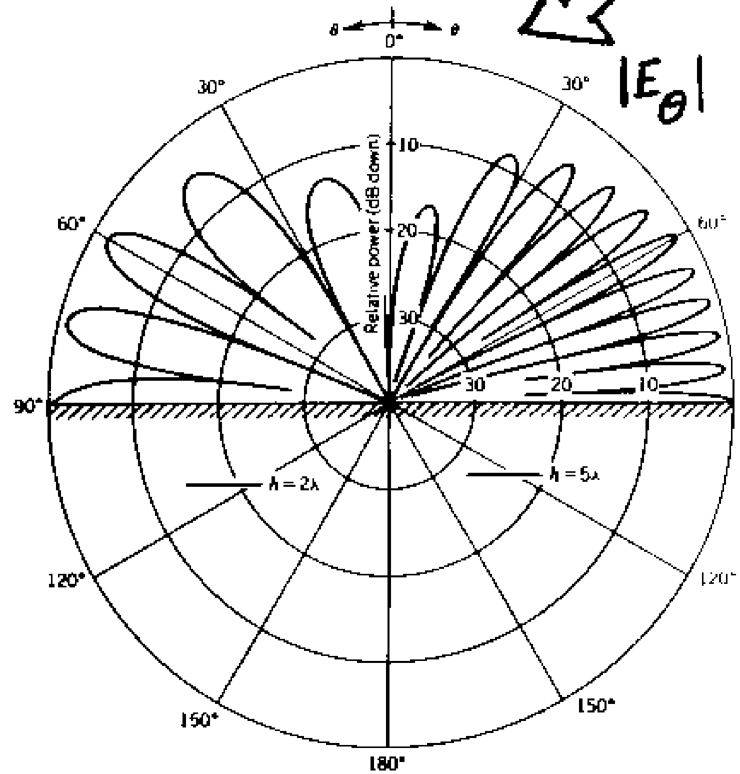


Elevation Pattern of a Vertical Infinitesimal Electric Dipole above an infinite Perfect Conductor Plane



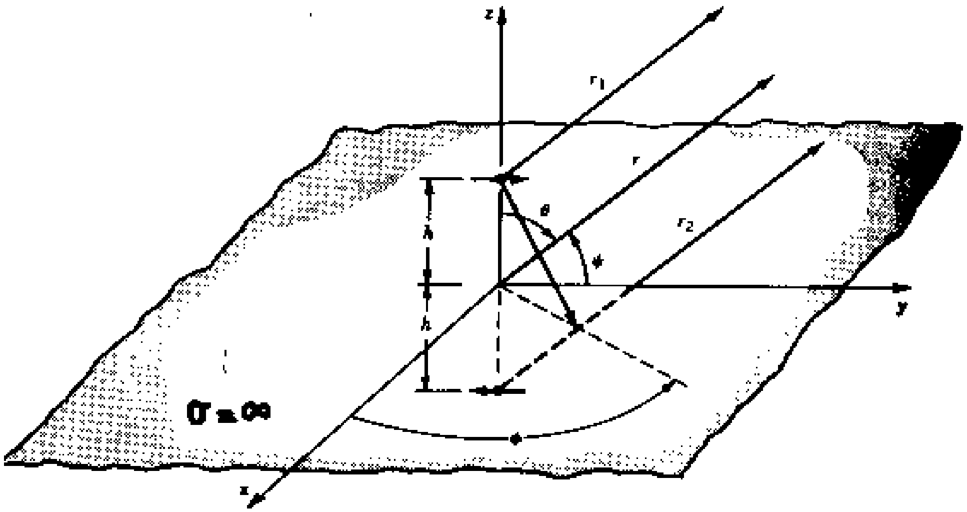
$$\begin{cases} E_\theta \simeq j\eta \dfrac{\beta I_0 \ell e^{-j\beta r}}{4\pi r} \underbrace{\sin\theta\left[2\cos(\beta h\cos\theta)\right]}_{z>0} \\ E_\theta = 0 \qquad z<0 \end{cases}$$

$|E_\theta|$



Application of the image theory provides an easy way to construct the radiated pattern.

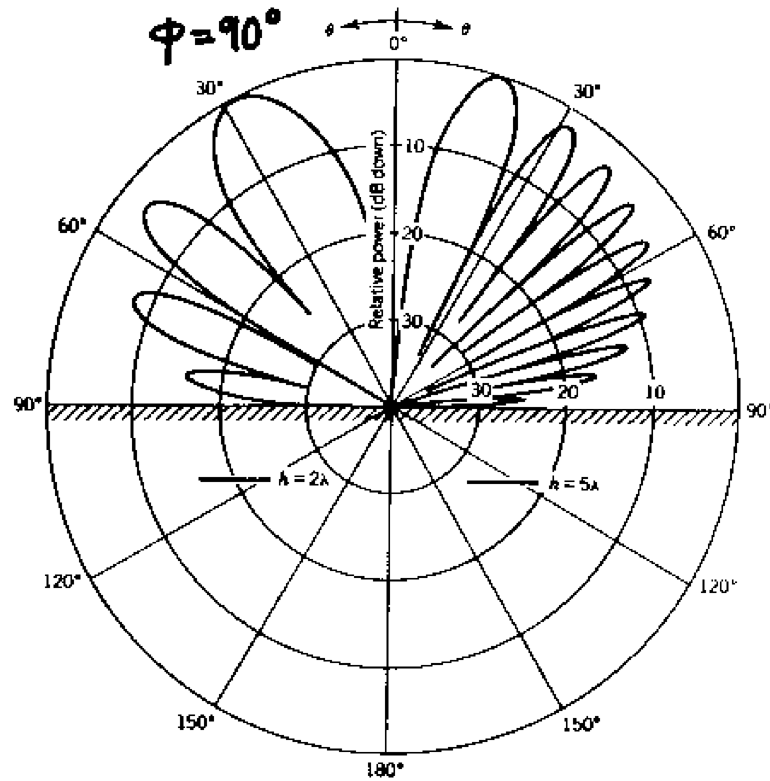
Elevation Pattern of a horizontal Infinitesimal Electric Dipole above an Infinite Perfect Conductor Plane



$$E_\psi = j\eta \frac{\beta I_0 l\, e^{-j\beta r}}{4\pi r} \sqrt{1-\sin^2\theta \sin^2\phi}$$

$$\cdot \left[2j \sin(\beta h \cos\theta)\right]$$

$\phi = 90°$



Field Equivalence Theorem (1)

Generalized boundary condition:

$$\hat{n}\cdot(\vec{D}_1-\vec{D}_2)=\rho_s \qquad C/m^2$$

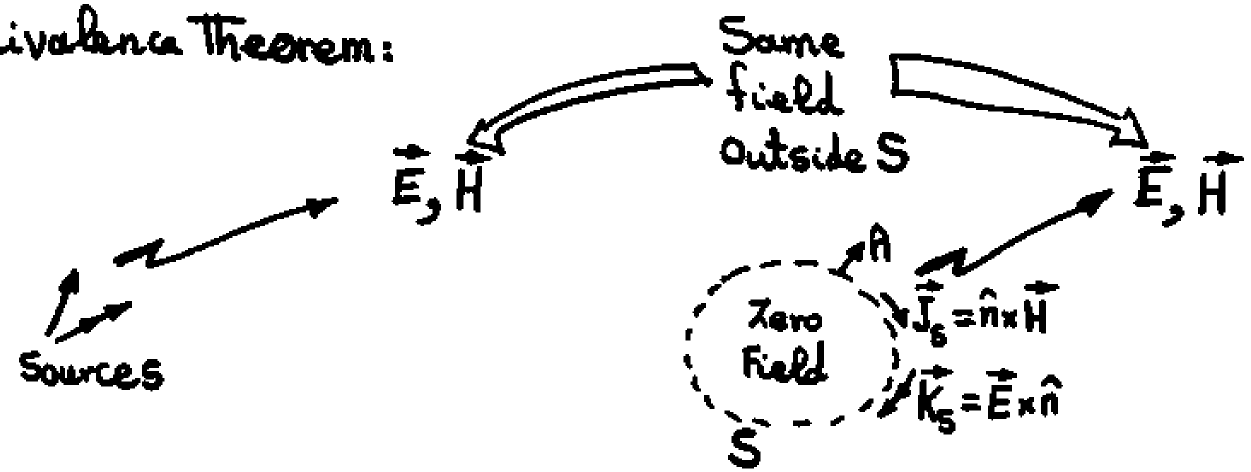
$$\hat{n}\cdot(\vec{B}_1-\vec{B}_2)=\rho_{ms} \qquad Wb/m^2$$

$$\hat{n}\times(\vec{E}_1-\vec{E}_2)=-\vec{M}_s \qquad V/m$$

$$\hat{n}\times(\vec{H}_1-\vec{H}_2)=\vec{J}_s \qquad A/m$$

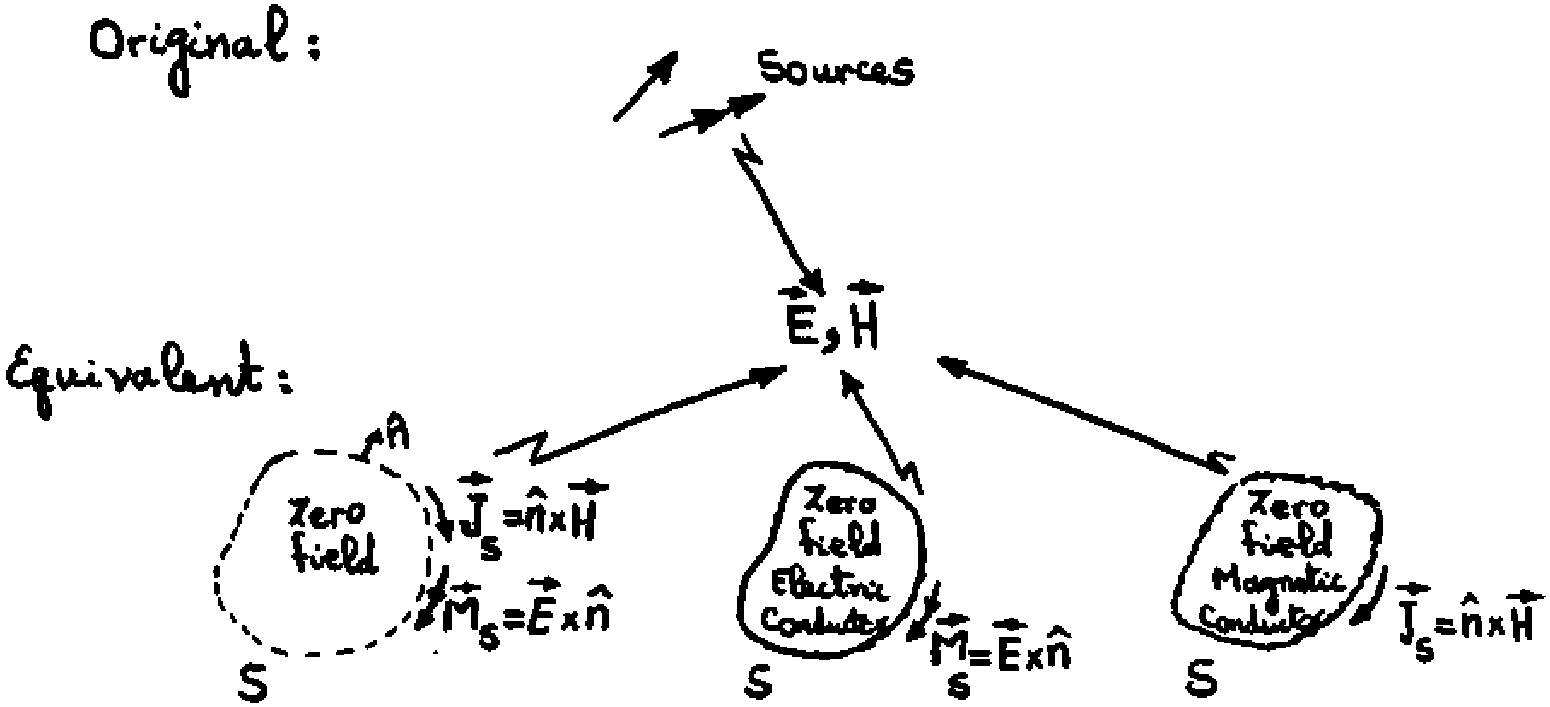
Fictitious

Equivalence Theorem:



Why ? : Because of Maxwell's Eqs., boundary conditions and uniqueness theorem

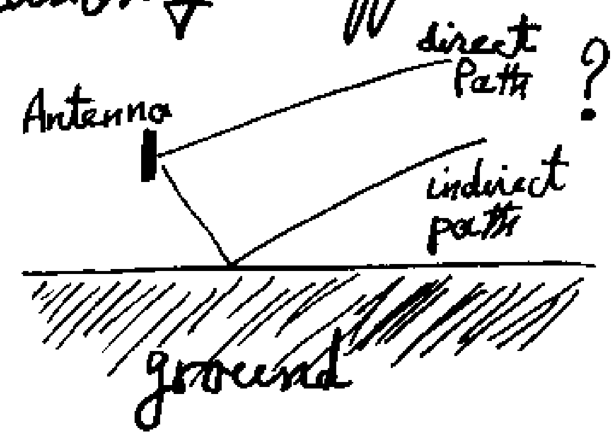
Field Equivalence theorem (2)



Equivalence theorem is used to determine radiation pattern of aperture antennas such as horns.

Effects of Real Earth on Antenna Patterns (1)
An Approximate Approach

There are many situations that antennas operate near the ground. It is therefore important to be able to estimate the effects of ground on the antenna performance. This is, in general, a very difficult electromagnetic problem to solve. However, application of an approximate method based on ground reflection coefficients allows one to obtain good estimates.

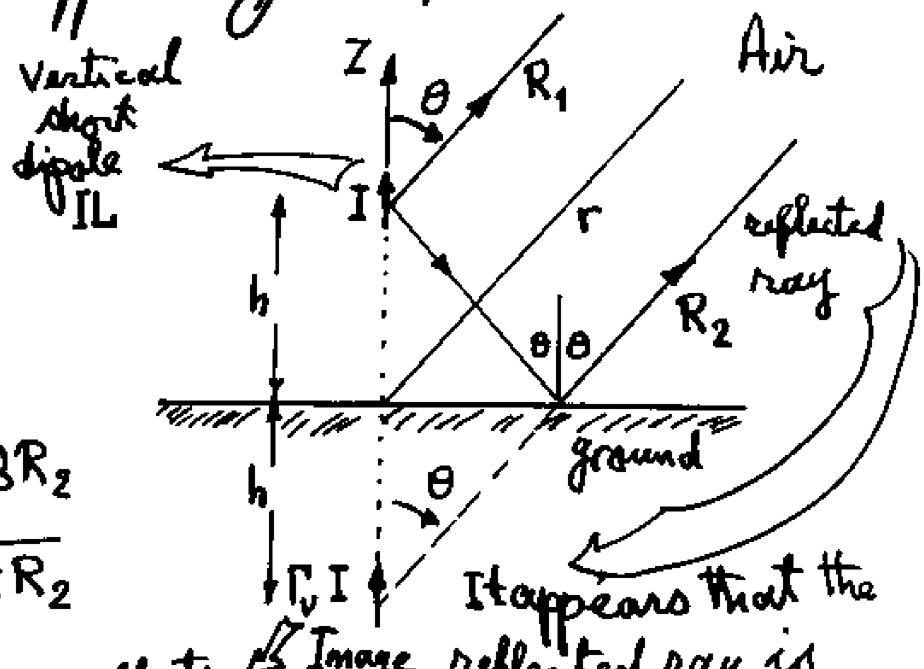


Effects of Real Earth on Antenna Patterns (2)

In many practical situations, one may approximate the presence of the ground as though it is radiating from the image point of the antenna affected by the reflection coefficients of the ground.

Dipole source: $E_\theta = j\omega\mu \sin\theta \cdot IL \cdot \dfrac{e^{-j\beta R_1}}{4\pi R_1}$

Image source: $E_\theta = j\omega\mu \sin\theta \cdot \Gamma_v IL \cdot \dfrac{e^{-j\beta R_2}}{4\pi R_2}$



reflection coefficient to account for ground

It appears that the reflected ray is emenated from the image source.

Effects of Real Earth on Antenna Patterns (3)

For far field condition: $R_1 \approx r - h\cos\theta$; $R_2 \approx r + h\cos\theta$

height of the antenna to the ground

Total far field: $$\boxed{E_\theta = j\omega\mu \frac{IL}{4\pi} \frac{e^{-j\beta r}}{r} \sin\theta \left(e^{j\beta h\cos\theta} + \Gamma_V e^{-j\beta h\cos\theta} \right)}$$

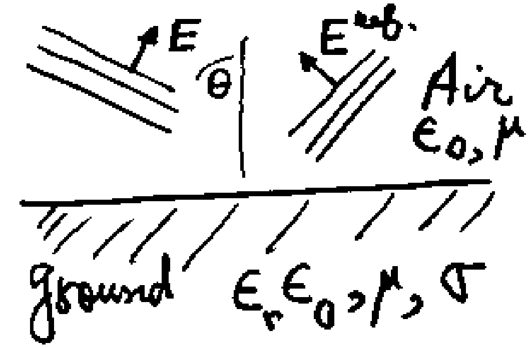
due to the source

due to its image

How to find Γ_V (this is a complex number): Γ_V is the reflection coefficient of a plane wave impinged upon the ground for the polarization in the plane of incidence.

Note: Similar results can be generated for the horizontal dipole antennas.

Effects of Real Earth on Antenna Patterns (4)



Γ_V can be found in closed form to:

related reflected field to the incident field

angle of incidence

$$\boxed{\Gamma_V = \frac{\epsilon_r' \cos\theta - \sqrt{\epsilon_r' - \sin^2\theta}}{\epsilon_r' \cos\theta + \sqrt{\epsilon_r' - \sin^2\theta}}}$$

this is a complex number. Γ_V has amplitude and phase.

where $\boxed{\epsilon_r' = \frac{\epsilon'}{\epsilon_0} = \epsilon_r - j\frac{\sigma}{\omega\epsilon_0}}$ $\omega = 2\pi f$ frequency

Example: Earth has average value of $\epsilon_r = 15$

Ground conductivity in U.S. vary from 10^{-3} to 3×10^{-2} S/m.

Effects of Real Earth on Antenna Patterns (5)

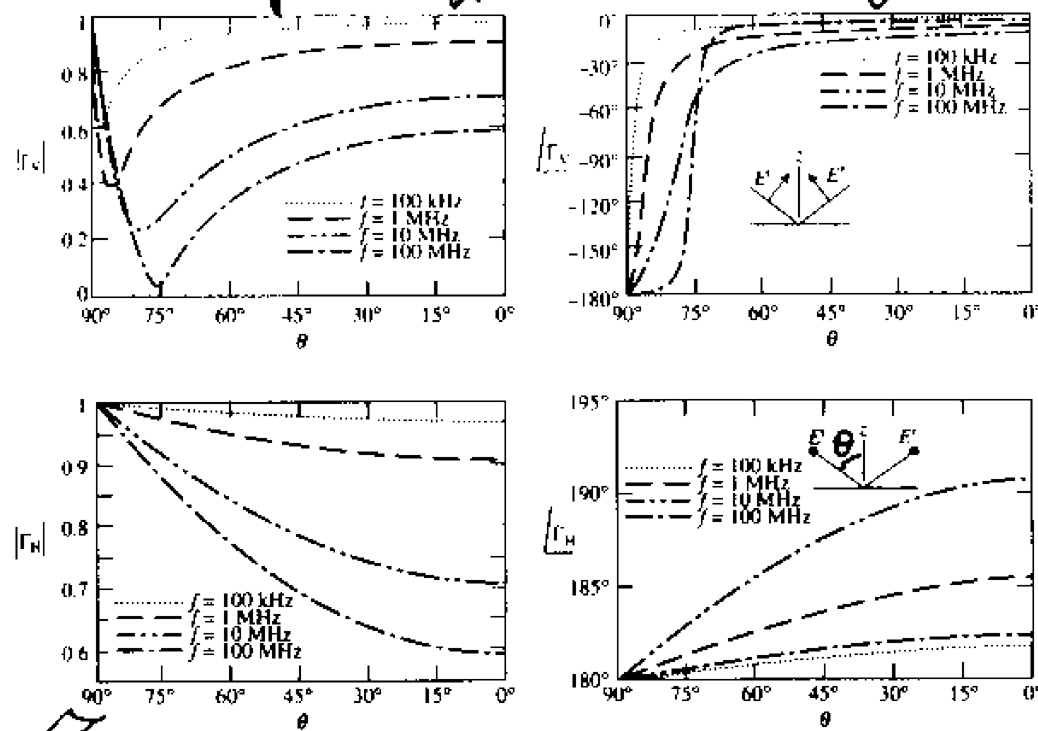
Amplitude of Γ Phase of Γ

Vertical

$$\Gamma_V = \frac{\epsilon_r' \cos\theta - \sqrt{\epsilon_r' - \sin^2\theta}}{\epsilon_r' \cos\theta + \sqrt{\epsilon_r' - \sin^2\theta}}$$

Horizontal

$$\Gamma_H = \frac{\cos\theta - \sqrt{\epsilon_r' - \sin^2\theta}}{\cos\theta + \sqrt{\epsilon_r' - \sin^2\theta}}$$



$\theta = 90^\circ$ In the direction of horizon

$\sigma = 12 \times 10^{-3}$ S/m ; $\epsilon_r = 15$

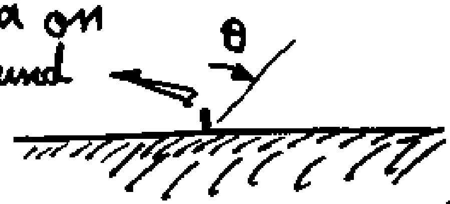
Note : $\epsilon_r' = \epsilon_r - j\dfrac{\sigma}{\omega\epsilon_0}$

$$\boxed{\frac{\sigma}{\omega\epsilon_0} = 18 \times 10^{3}\,\frac{\sigma}{f\,(\text{in MHz})}}$$

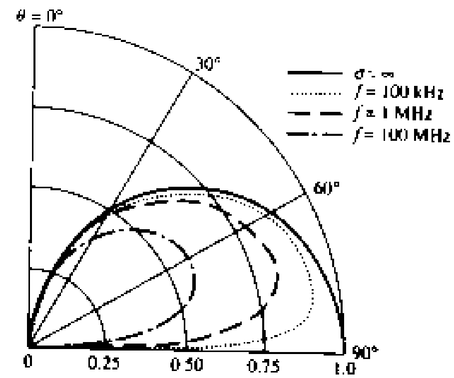
freq.

Effects of Real Earth on Antenna Patterns (6)

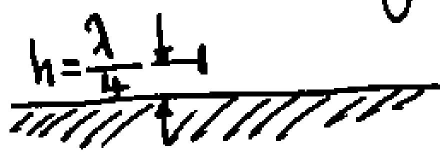
Antenna on the ground

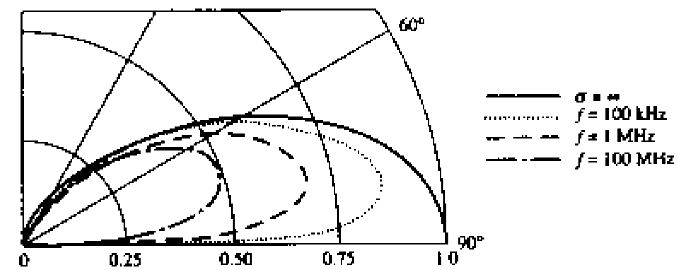


$\epsilon_r = 15; \ \sigma = 12 \times 10^{-3}$ S/m

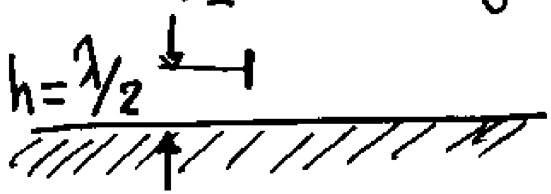


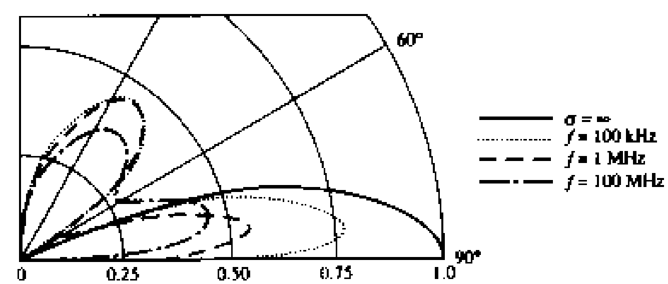
Antenna $\lambda/4$ above the ground





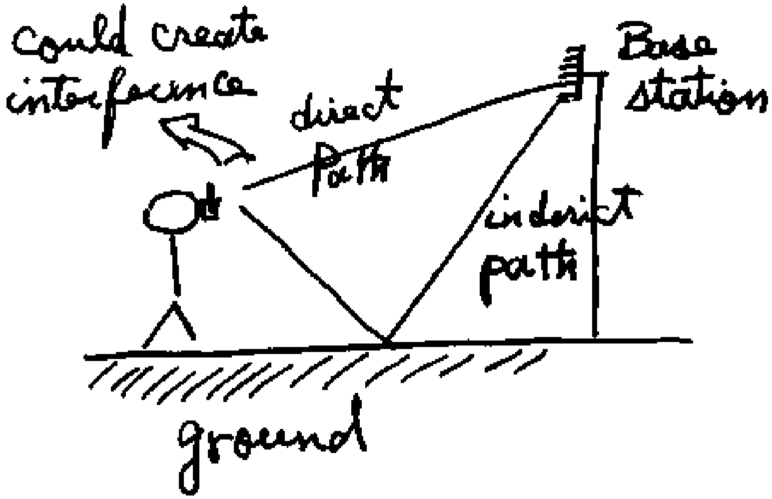
Antenna $\lambda/2$ above the ground

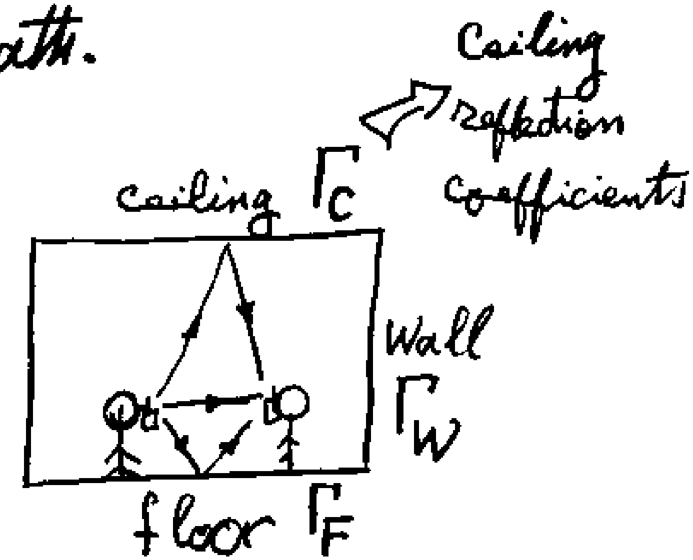




Multipath Effects

Similar approximate techniques can be used to estimate the effects of multipath.



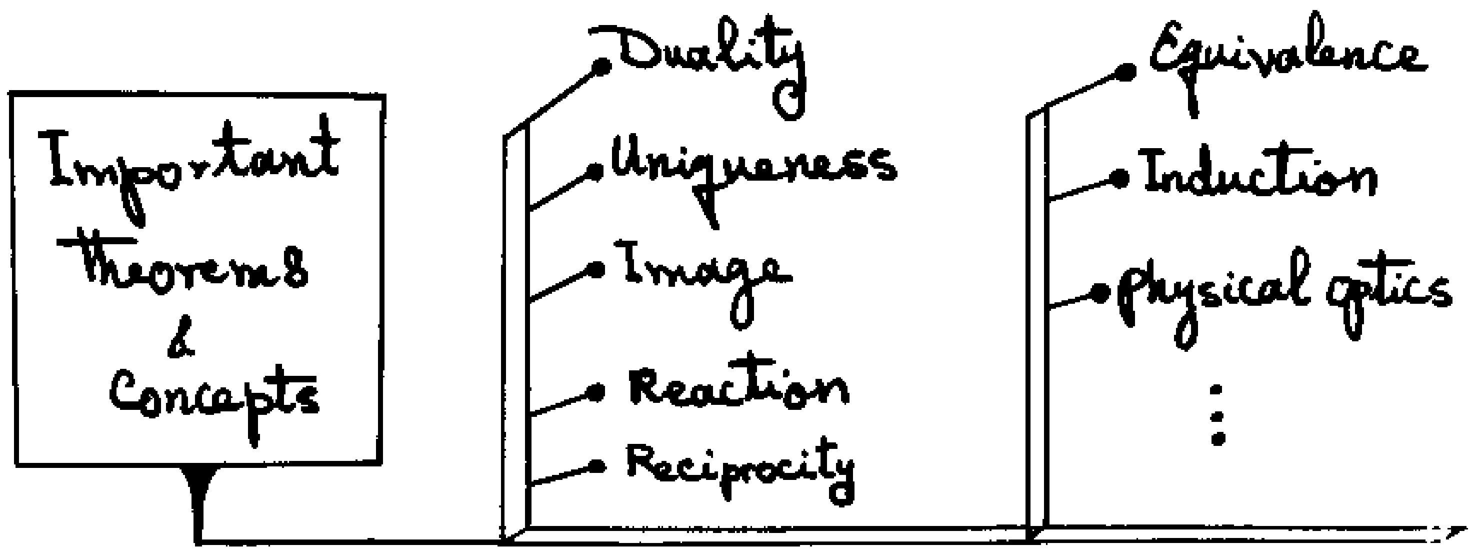


Very complex interference situation can exist.

- There are also multi-path effects due to sharp edges and corners.

Theorems & Concepts

There are important Theorems and concepts which facilitate applications and understanding of electromagnetic generation, propagation, radiation, scattering, reception etc.



Duality in E.M.

Dual equations for electric (J) and magnetic (M) current sources

Electric sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
$\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A$	$\nabla \times \mathbf{H}_F = j\omega\varepsilon\mathbf{E}_F$
$\nabla \times \mathbf{H}_A = \mathbf{J} + j\omega\varepsilon\mathbf{E}_A$	$-\nabla \times \mathbf{E}_F = \mathbf{M} + j\omega\mu\mathbf{H}_F$
$\nabla^2\mathbf{A} + \beta^2\mathbf{A} = -\mu\mathbf{J}$	$\nabla^2\mathbf{F} + \beta^2\mathbf{F} = -\varepsilon\mathbf{M}$
$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-j\beta R}}{R}\,dv'$	$\mathbf{F} = \frac{\varepsilon}{4\pi}\iiint_V \mathbf{M}\frac{e^{-j\beta R}}{R}\,dv'$
$\mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A}$	$\mathbf{E}_F = -\frac{1}{\varepsilon}\nabla \times \mathbf{F}$
$\mathbf{E}_A = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{A})$	$\mathbf{H}_F = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{F})$

Dual quantities for electric (J) and magnetic (M) current sources

Electric sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
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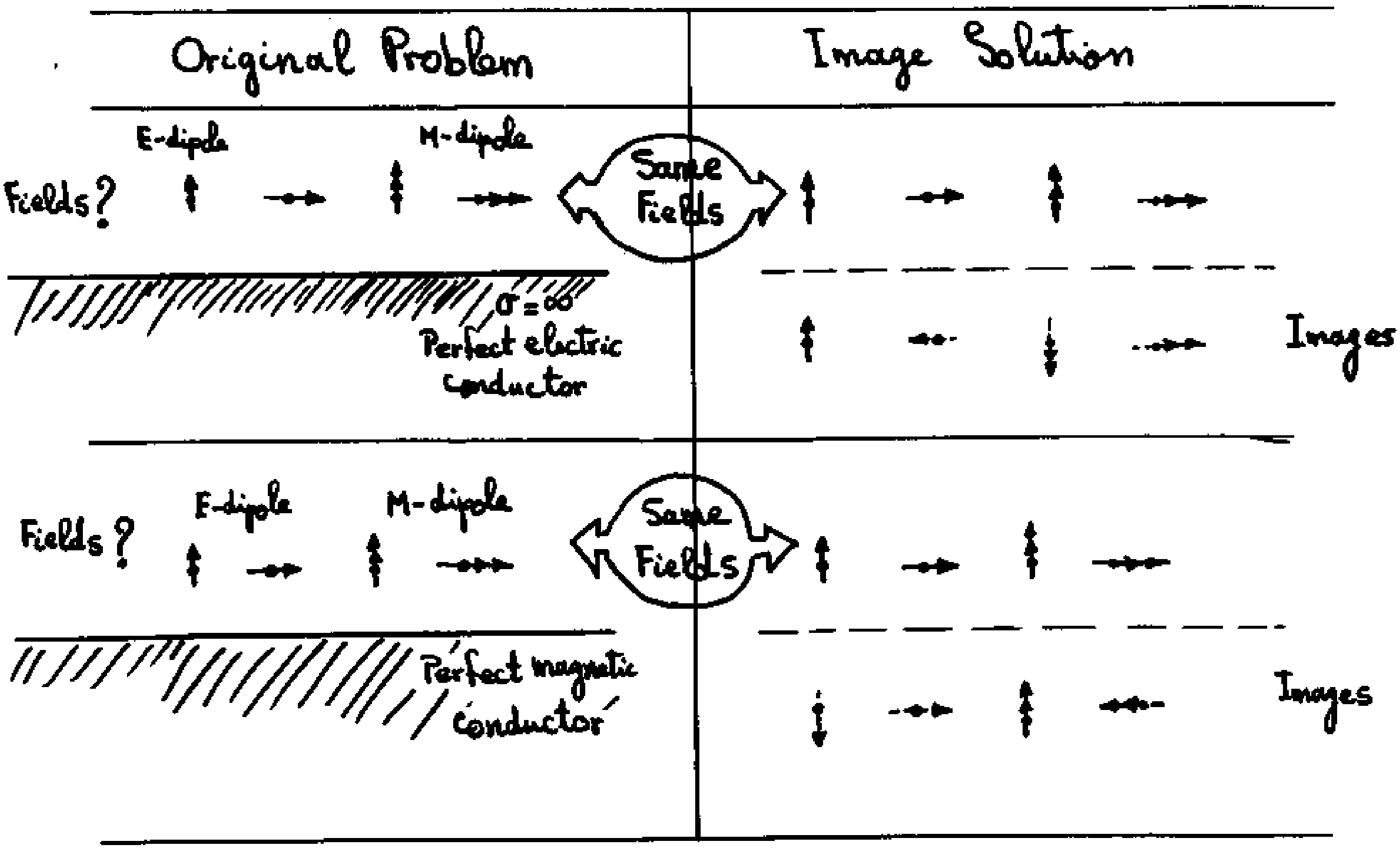
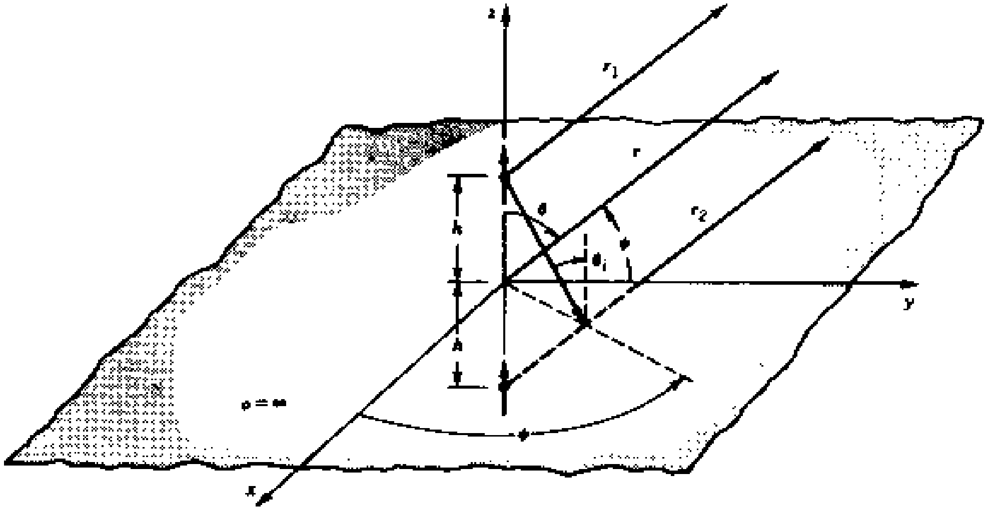
$\mathbf{H}_A$	$-\mathbf{E}_F$
$\mathbf{J}$	$\mathbf{M}$
$\mathbf{A}$	$\mathbf{F}$
ε	μ
μ	ε
β	β
η	$1/\eta$
$1/\eta$	η

Image Theory

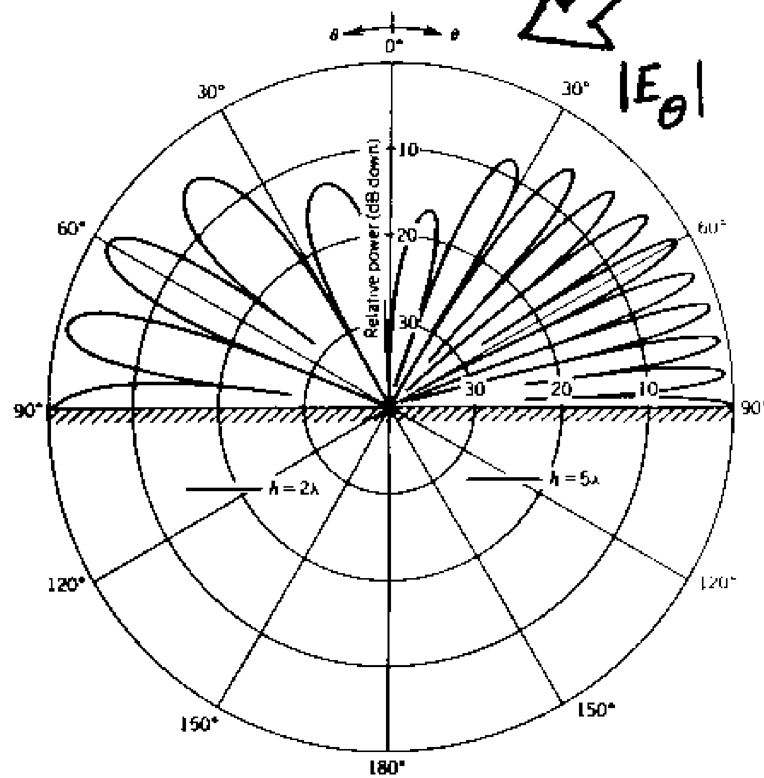


Elevation Pattern of a Vertical Infinitesimal Electric Dipole above an infinite Perfect Conductor Plane



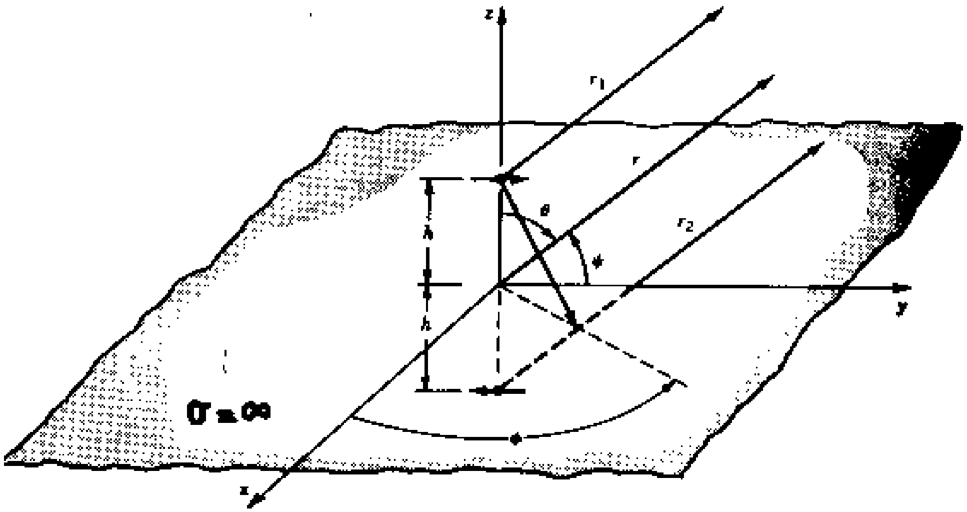
$$\begin{cases} E_\theta \simeq j\eta \dfrac{\beta I_0 \ell e^{-j\beta r}}{4\pi r} \sin\theta\left[2\cos(\beta h\cos\theta)\right] & z>0 \\ E_\theta = 0 & z<0 \end{cases}$$

$|E_\theta|$



Application of the image theory provides an easy way to construct the radiated pattern.

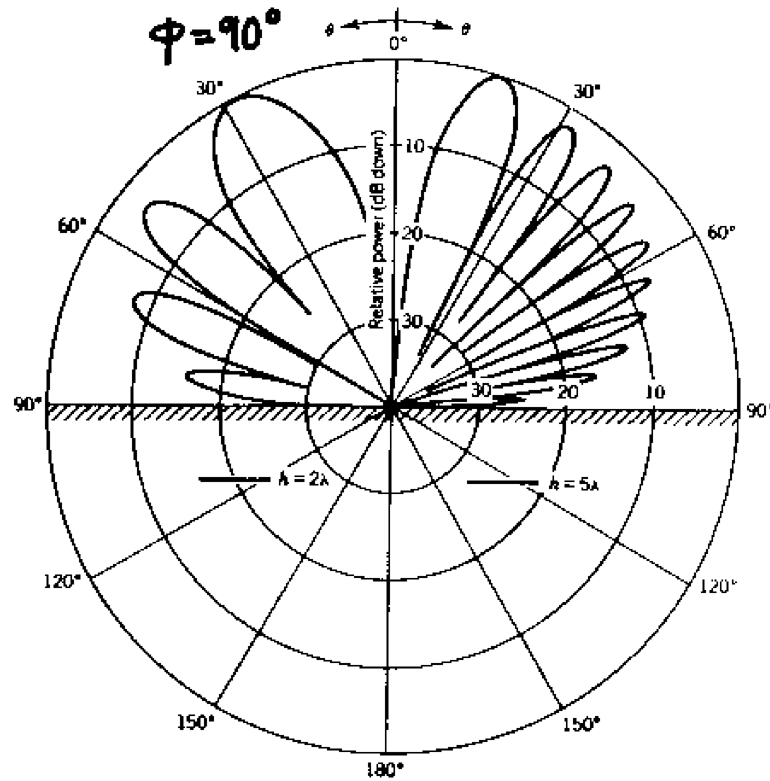
Elevation Pattern of a horizontal Infinitesimal Electric Dipole above an Infinite Perfect Conductor Plane



$$E_\psi = j\eta \frac{\beta I_0 l e^{-j\beta r}}{4\pi r} \sqrt{1-\sin^2\theta \sin^2\phi}$$

$$\cdot \left[2j \sin(\beta h \cos\theta)\right]$$

$\phi = 90°$



Field Equivalence Theorem (1)

Generalized boundary condition :

$$\hat{n}\cdot(\vec{D}_1-\vec{D}_2)=\rho_s \qquad C/m^2$$

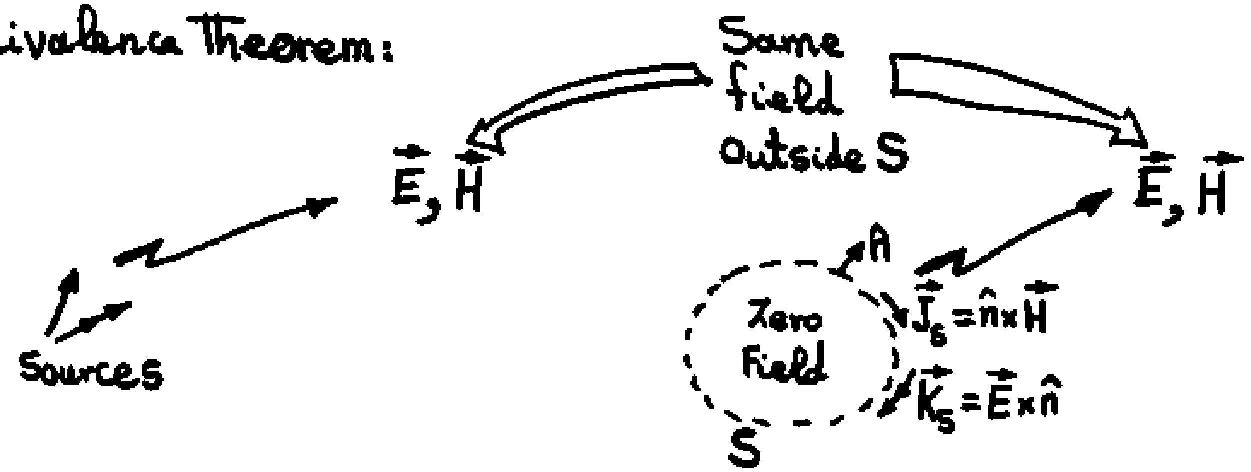
$$\hat{n}\cdot(\vec{B}_1-\vec{B}_2)=\rho_{ms} \qquad Wb/m^2$$

$$\hat{n}\times(\vec{E}_1-\vec{E}_2)=-\vec{M}_s \qquad V/m$$

$$\hat{n}\times(\vec{H}_1-\vec{H}_2)=\vec{J}_s \qquad A/m$$

Fictitious

Equivalence Theorem:

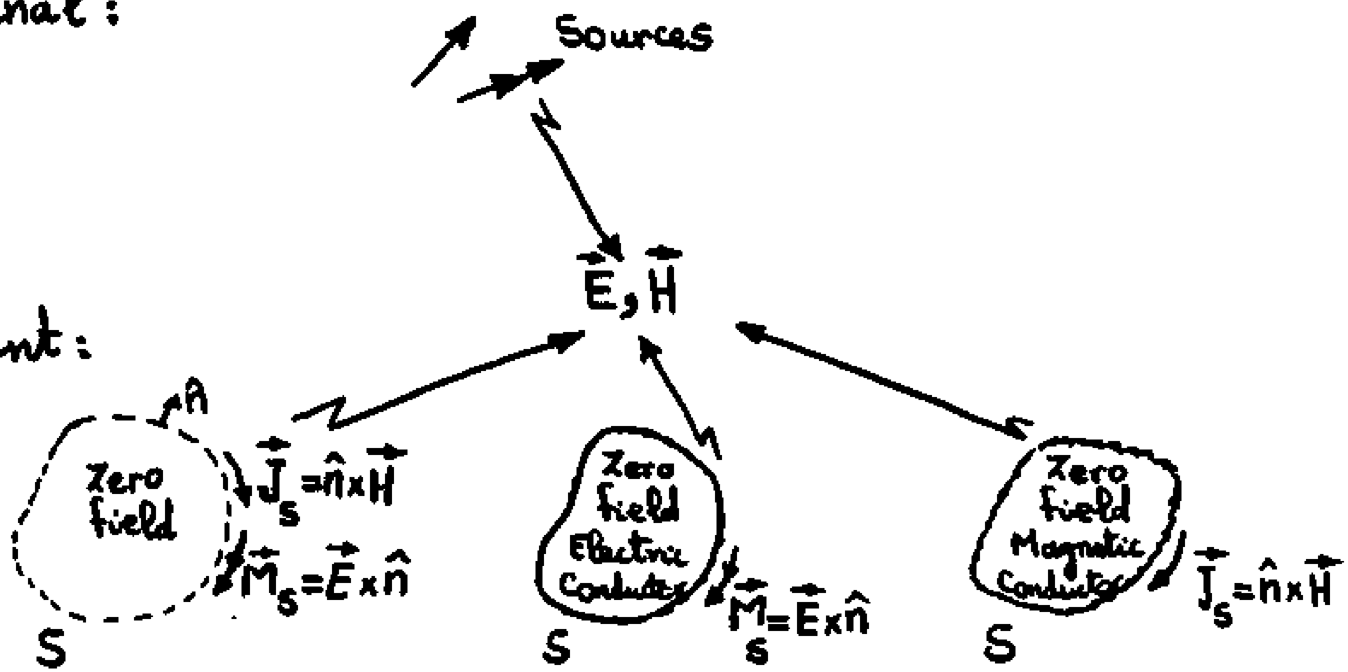


Why ? : Because of Maxwell's Eqs., boundary conditions and uniqueness theorems

Field Equivalence theorem (2)

Original:

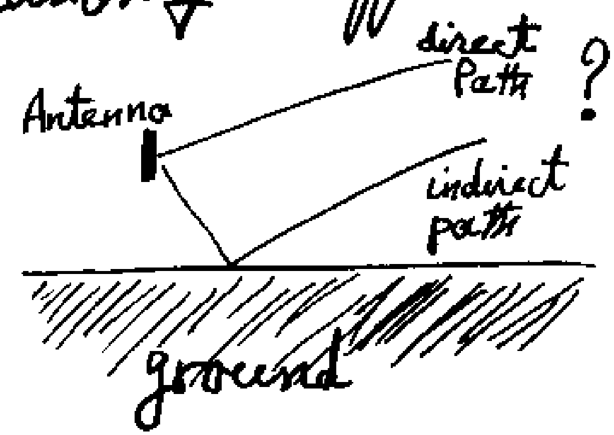
Equivalent:



Equivalence theorem is used to determine radiation pattern of aperture antennas such as horns.

Effects of Real Earth on Antenna Patterns (1)
An Approximate Approach

There are many situations that antennas operate near the ground. It is therefore important to be able to estimate the effects of ground on the antenna performance. This is, in general, a very difficult electromagnetic problem to solve. However, application of an approximate method based on ground reflection coefficients allows one to obtain good estimates.

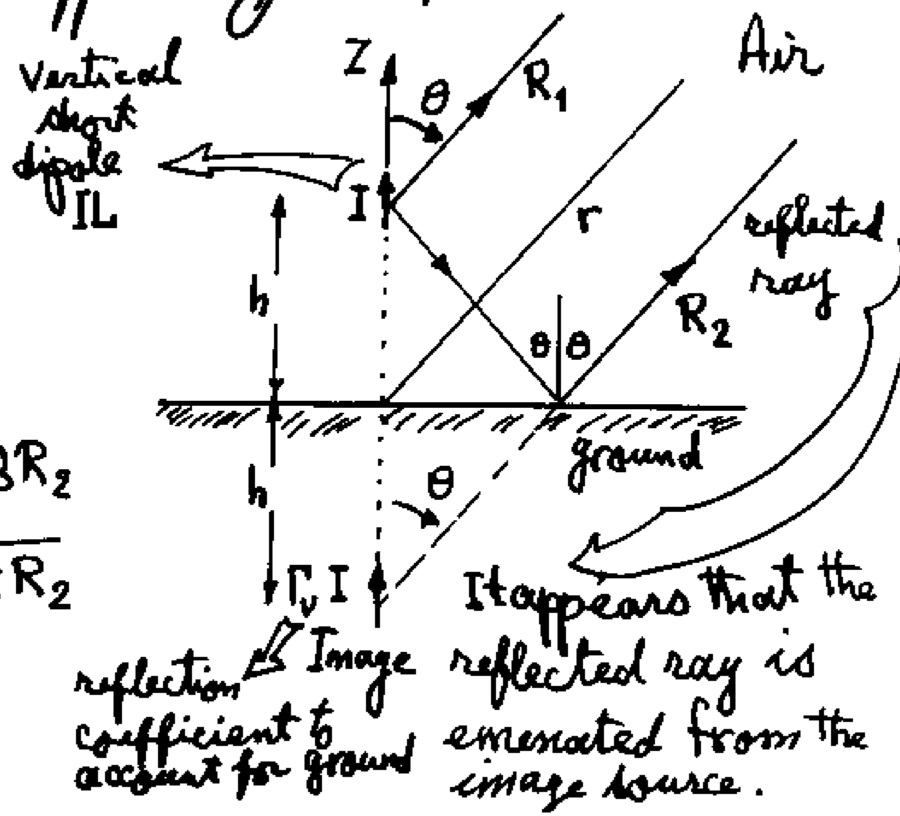


Effects of Real Earth on Antenna Patterns (2)

In many practical situations, one may approximate the presence of the ground as though it is radiating from the image point of the antenna affected by the reflection coefficients of the ground.

Dipole source: $E_\theta = j\omega\mu\sin\theta \cdot IL \cdot \dfrac{e^{-j\beta R_1}}{4\pi R_1}$

Image source: $E_\theta = j\omega\mu\sin\theta \cdot \Gamma_v IL \cdot \dfrac{e^{-j\beta R_2}}{4\pi R_2}$



Effects of Real Earth on Antenna Patterns (3)

For far field condition: $R_1 \approx r - h\cos\theta$; $R_2 \approx r + \underbrace{h}_{\text{height of the antenna to the ground}}\cos\theta$

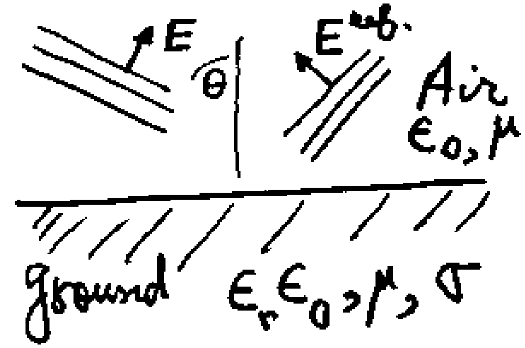
Total far field:

$$\boxed{E_\theta = j\omega\mu \frac{IL}{4\pi} \frac{e^{-j\beta r}}{r} \sin\theta \left(\underbrace{e^{j\beta h\cos\theta}}_{\text{due to the source}} + \underbrace{\Gamma_V e^{-j\beta h\cos\theta}}_{\text{due to its image}} \right)}$$

How to find Γ_V (this is a complex number): Γ_V is the reflection coefficient of a plane wave impinged upon the ground for the polarization in the plane of incidence.

Note: Similar results can be generated for the horizontal dipole antennas.

Effects of Real Earth on Antenna Patterns (4)



Γ_V can be found in closed form to:

related reflected field to the incident field

angle of incidence

$$\boxed{\Gamma_V = \frac{\epsilon_r' \cos\theta - \sqrt{\epsilon_r' - \sin^2\theta}}{\epsilon_r' \cos\theta + \sqrt{\epsilon_r' - \sin^2\theta}}}$$

this is a complex number. Γ_V has amplitude and phase.

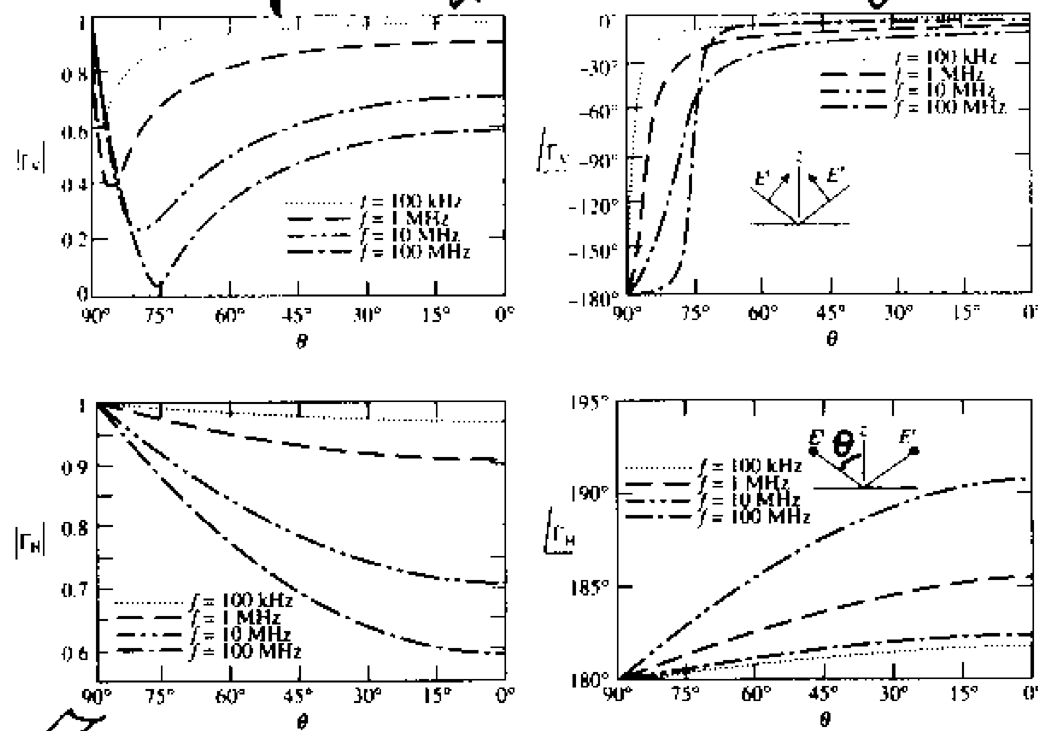
where $\boxed{\epsilon_r' = \frac{\epsilon'}{\epsilon_0} = \epsilon_r - j\frac{\sigma}{\omega\epsilon_0}}$ $\quad \omega = 2\pi f$ → frequency

Example: Earth has average value of $\epsilon_r = 15$

Ground conductivity in U.S. vary from 10^{-3} to 3×10^{-2} S/m.

Effects of Real Earth on Antenna Patterns (5)

Amplitude of Γ — Phase of Γ



Vertical

$$\Gamma_V = \frac{\epsilon_r' \cos\theta - \sqrt{\epsilon_r' - \sin^2\theta}}{\epsilon_r' \cos\theta + \sqrt{\epsilon_r' - \sin^2\theta}}$$

Horizontal

$$\Gamma_H = \frac{\cos\theta - \sqrt{\epsilon_r' - \sin^2\theta}}{\cos\theta + \sqrt{\epsilon_r' - \sin^2\theta}}$$

$\theta = 90^\circ$ In the direction of horizon

$\sigma = 12 \times 10^{-3}$ S/m ; $\epsilon_r = 15$

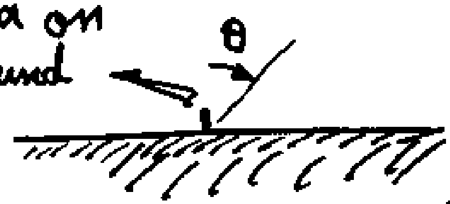
Note : $\epsilon_r' = \epsilon_r - j\dfrac{\sigma}{\omega\epsilon_0}$

$$\boxed{\frac{\sigma}{\omega\epsilon_0} = 18 \times 10^3 \, \frac{\sigma}{f\,(\text{in MHz})}}$$

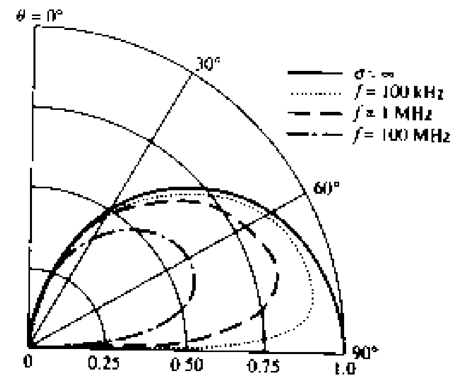
freq.

Effects of Real Earth on Antenna Patterns (6)

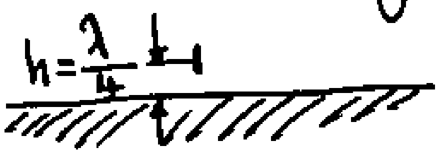
Antenna on the ground

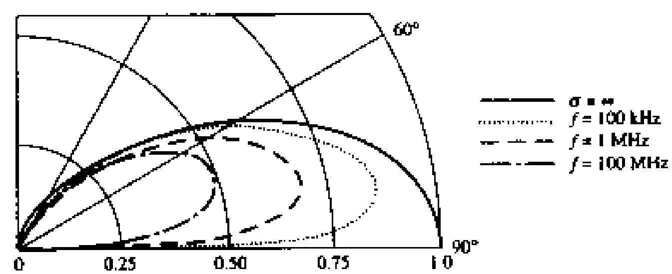


$\epsilon_r = 15;\ \sigma = 12 \times 10^{-3}\ \mathrm{S/m}$

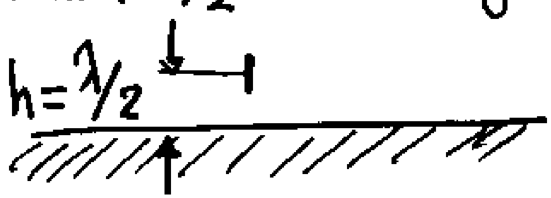


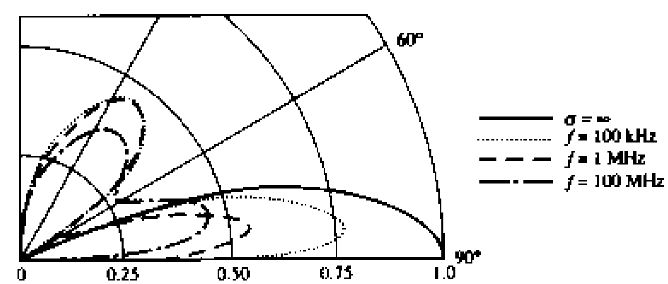
Antenna $\lambda/4$ above the ground





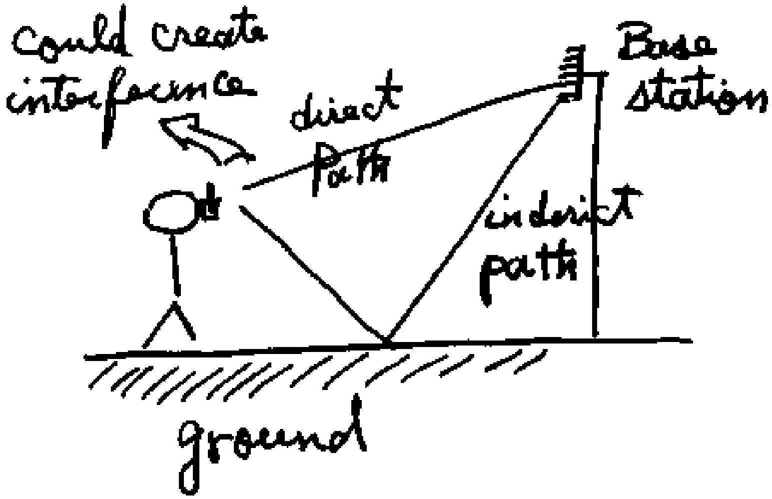
Antenna $\lambda/2$ above the ground

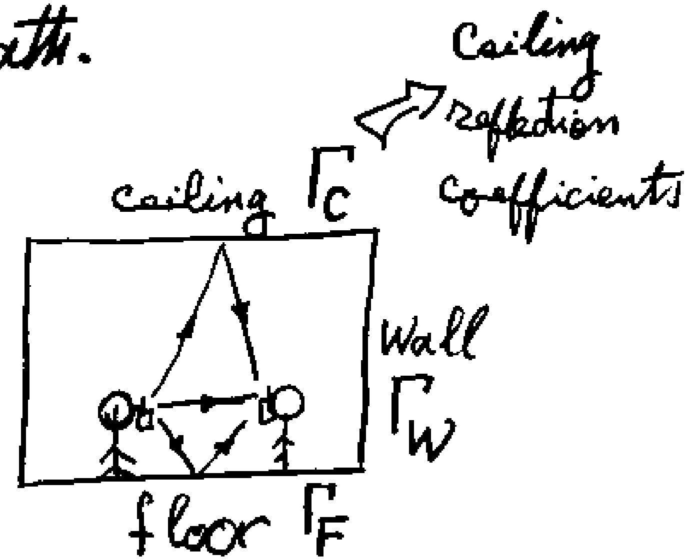




Multipath Effects

Similar approximate techniques can be used to estimate the effects of multipath.





Very complex interference situation can exist.

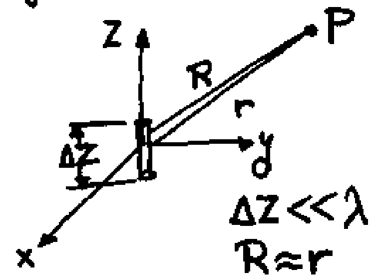
- There are also multi-path effects due to sharp edges and corners.

The Ideal Dipole

(Hertzian electric dipole, infinitesimal dipole, and doublet)

1. Write the expression of the current

$$\vec{J} = I\,\delta(x')\,\delta(y')\,\hat{z} = I\Delta z\,\overbrace{\delta(x')\,\delta(y')\,\delta(z')}^{\delta(\vec{r}')}\,\hat{z}$$



$\Delta z \ll \lambda$

$R \approx r$

2. Construct vector potential $\vec{A}$

$$\vec{A} = \int_V \mu \vec{J}\,\frac{e^{-j\beta R}}{4\pi R}\,dv' = \hat{z}\mu I \int \Delta z\,\delta(\vec{r}')\,\frac{e^{-j\beta|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\,dv' = \mu I \Delta z\,\frac{e^{-j\beta r}}{4\pi r}\,\hat{z}$$

$R \Rightarrow |\vec{r}-\vec{r}'|$

or $\boxed{\vec{A} = \mu I \Delta z\,\frac{e^{-j\beta r}}{4\pi r}\,\hat{z}}$

3. Find $\vec{H}$ field: $\vec{H} = \frac{1}{\mu}\nabla\times\vec{A} \Rightarrow \boxed{\vec{H} = \frac{I\Delta z}{4\pi}\,j\beta\left(1+\frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\phi}}$

4. Find $\vec{E}$ field: $\vec{E} = \frac{1}{j\omega\epsilon}\nabla\times\vec{H} \Rightarrow$

$$\boxed{\begin{aligned}\vec{E} = {} & \frac{I\Delta z}{4\pi}\,j\omega\mu\left[1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\theta} \\ & + \frac{I\Delta z}{2\pi}\,\eta\left[\frac{1}{r}-j\frac{1}{\beta r^2}\right]\frac{e^{-j\beta r}}{r}\cos\theta\,\hat{r}\end{aligned}}$$

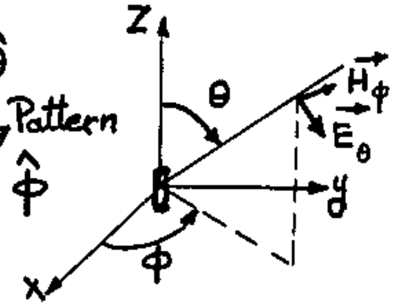
The Ideal Dipole: Radiated Field

Radiation is characterized by field components decaying as "$\frac{1}{r}$".

Radiated field of an ideal dipole:

$$\begin{cases} \vec{E} = j\omega\mu I \Delta Z \dfrac{e^{-j\beta r}}{4\pi r} \sin\theta \, \hat{\theta} \\ \vec{H} = j\omega\sqrt{\mu\epsilon}\, I \Delta Z \dfrac{e^{-j\beta r}}{4\pi r} \sin\theta \, \hat{\phi} \end{cases}$$

Pattern



Some very important observations for far fields:

① $\vec{E}$ & $\vec{H}$ fields are in phase

② $\vec{E}$ & $\vec{H}$ fields are transverse to the direction of propagation (i.e. no radial component)

③ the ratio of $|\vec{E}|/|\vec{H}| = \sqrt{\frac{\mu}{\epsilon}} = \eta$ free space impedance

④ the angular dependence defines radiation pattern.

⑤ $\vec{E}$, $\vec{H}$ and $\hat{r}$ are orthogonal to each other.

knowing $\vec{E}$ one also knows $\vec{H}$

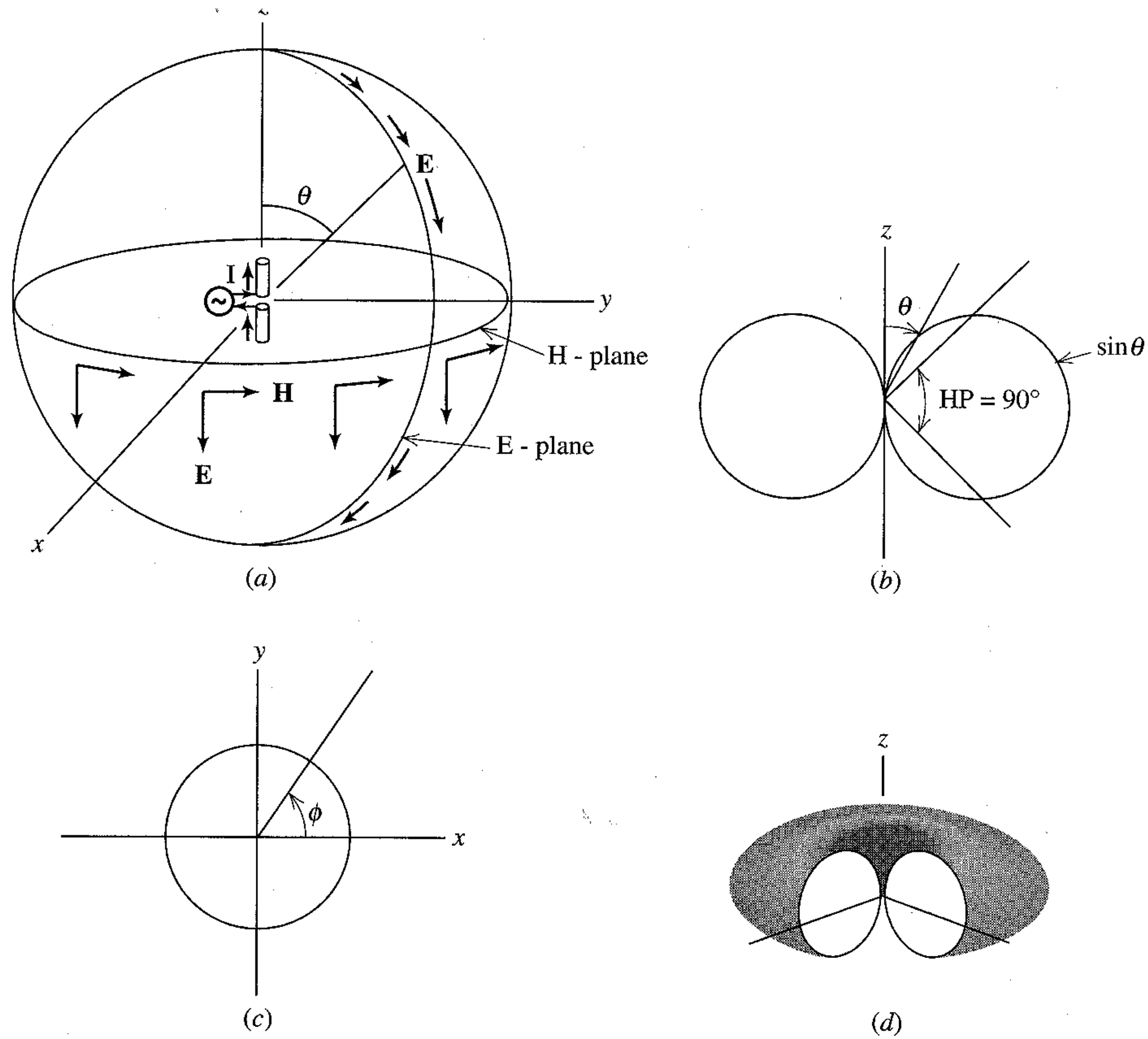
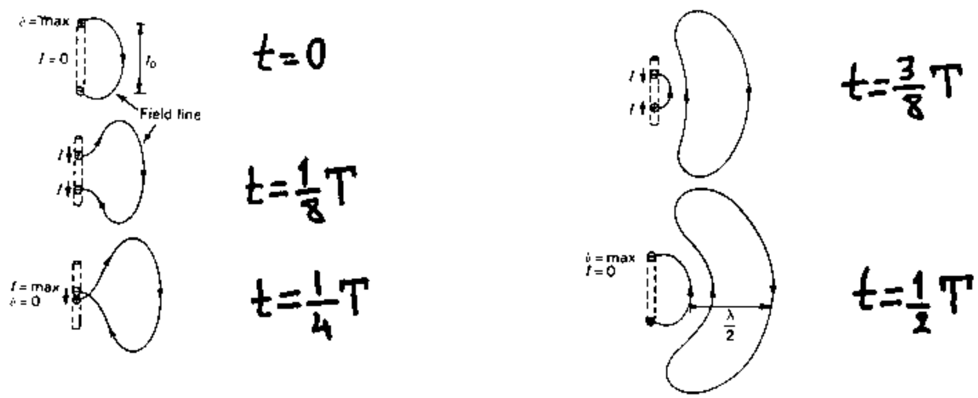


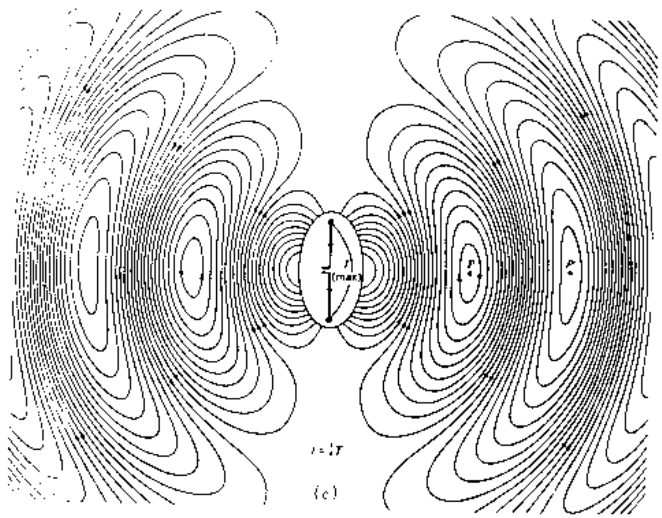
Figure 1-10 Radiation from an ideal dipole. (*a*) Field components. (*b*) *E*-plane radiation pattern polar plot of $|E_\theta|$ or $|H_\phi|$. (*c*) *H*-plane radiation pattern polar plot of $|E_\theta|$ or $|H_\phi|$. (*d*) Three-dimensional plot of radiation pattern.

Propagation of an Electric Field line and its Detachment (Radiation) from the Dipole.

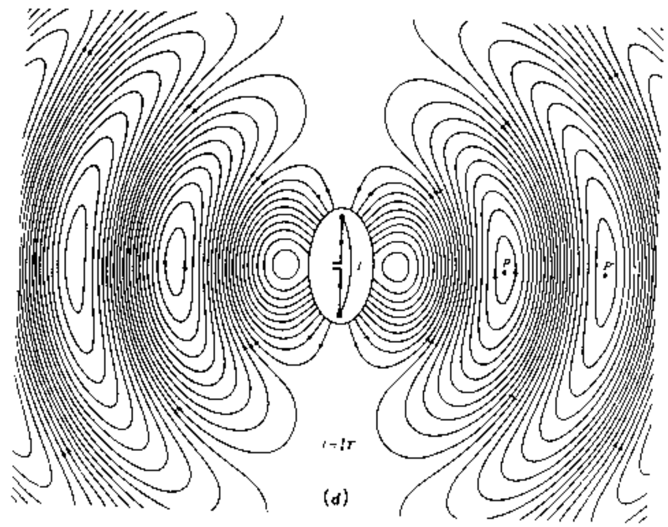


T : period of oscillation.

E field lines of a $\lambda/2$ antenna

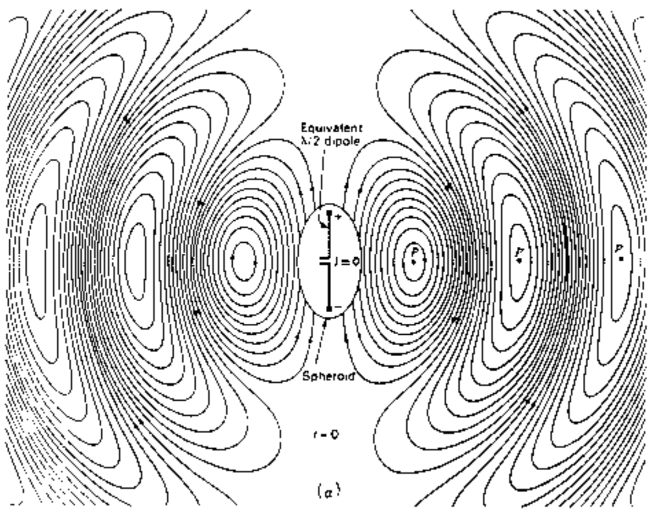


$t = \frac{1}{4} T$

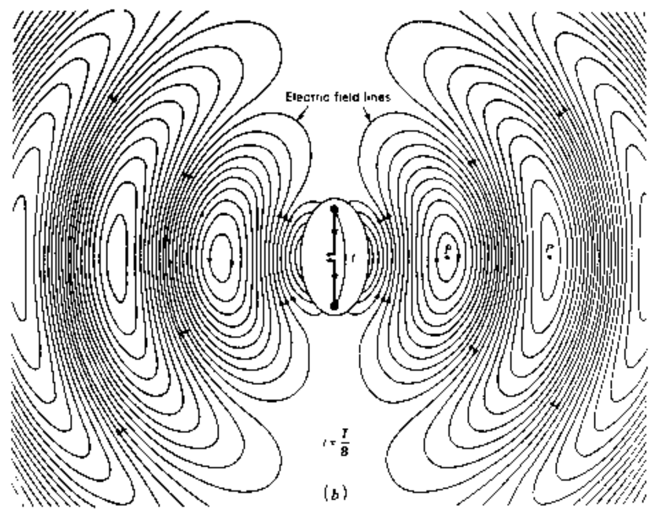


$t = \frac{3}{8} T$

E field lines of a $\lambda/2$ antenna



$t=0$

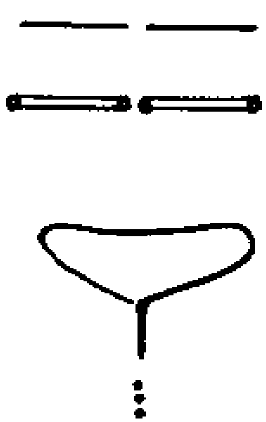


$t=\frac{1}{8}T$

Resonant Antennas: Wires (1)

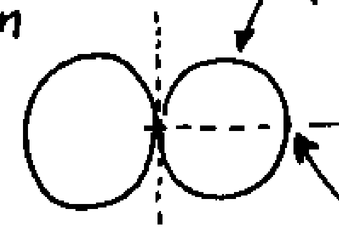
Definition: Antennas whose dimensions are comparable to the wavelength of the operating frequency are called "resonant antennas".

Wire Antennas: Wire antennas are the oldest and still the most prevalent of all antenna forms. Just about every imaginable shape and configuration of wires has a useful antenna applications. Wire antenna can be made from either solid wire or tubular conductors. They are relatively simple in concept, easy to construct, and inexpensive.



Resonant Antennas: Wires (2)

Parameters of interests:



...

Observation: All of the above parameters can be obtained if one knows the current distribution on the antenna.

Various methods:

- simple models
- integral equation & approximate analytical solutions
- integral equation & Method of moments
- ...

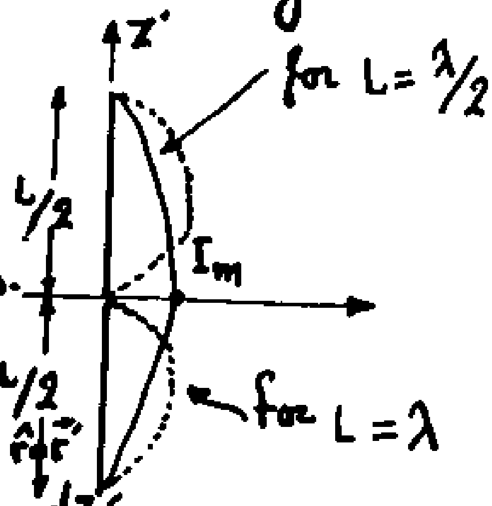
Resonant Antennas: Wires (3)

Current density: $\vec{J}(\vec{r}') = \underbrace{I(z')}_{A}\,\underbrace{\delta(x')}_{\frac{1}{m}}\,\underbrace{\delta(y')}_{\frac{1}{m}}\,\hat{z}$ $\quad A/m^2$ $\Rightarrow$ dipole along z axis

A good approximation for the current: $I(z') = I_m \sin[\beta(\frac{L}{2} - |z'|)$ $\Rightarrow$ dipole length

Note: • Current goes to zero at the ends.

• For different length dipoles, current distribution takes various distributions.



For far field computation:

$$\vec{A} = \mu \frac{e^{-j\beta r}}{4\pi r}\int_V \vec{J}(r')\, e^{j\beta \hat{r}\cdot\vec{r}'}\, dv' = \mu \frac{e^{-j\beta r}}{4\pi r}\int_{-L/2}^{L/2} I(z')\, e^{j\beta \hat{r}\cdot\vec{r}'}\, dz'$$

Resonant Antennas: Wires (4)

Recall: For $\hat{z}$ oriented linear current $\hat{r}\cdot\vec{r}\,' = z'\cos\theta$

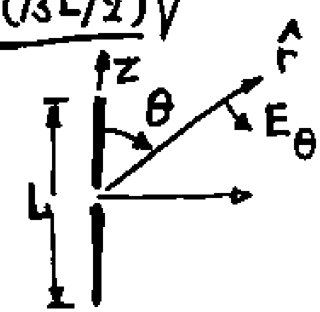
Far Field: $$E_\theta = j\omega A_\theta = j\omega\sin\theta\, A_z = j\omega\mu\sin\theta\,\frac{e^{-j\beta r}}{4\pi r}\int_{-L/2}^{L/2} I(z')\,e^{j\beta z'\cos\theta}\,dz'$$

We need to evaluate this.

Let: $$f_{un} = \int_{-L/2}^{0} I_m \sin[\beta(\tfrac{L}{2}+z')]\,e^{j\beta z'\cos\theta}\,dz' + \int_{0}^{L/2} I_m \sin[\beta(\tfrac{L}{2}-z')]\,e^{j\beta z'\cos\theta}\,dz'$$

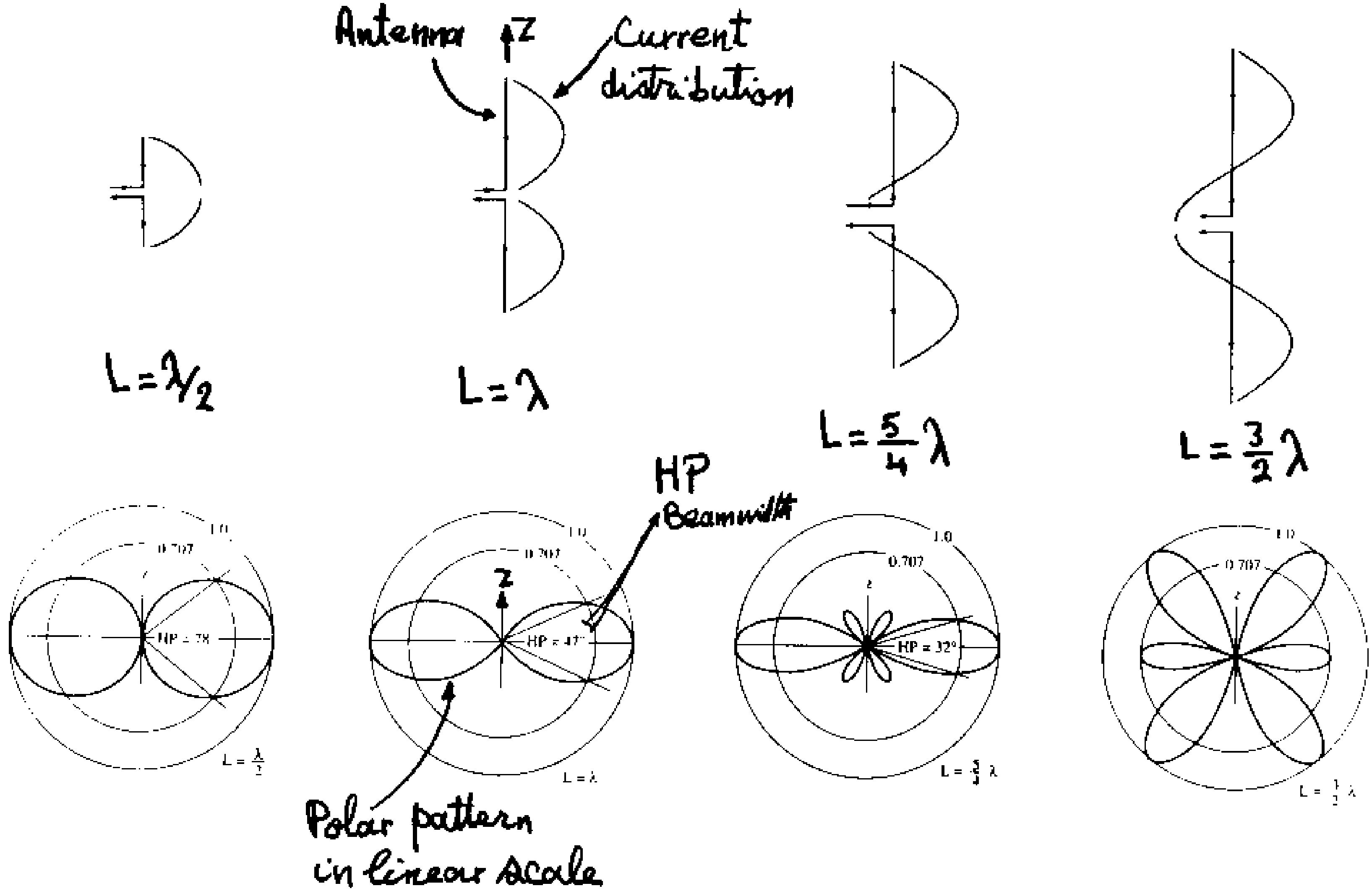
After some manipulations: $$\boxed{E_\theta = j\underbrace{\eta}_{\sqrt{\frac{\mu}{\epsilon}}\ \text{(free space impedance)}}\,\frac{e^{-j\beta r}}{4\pi r}\, I_m\, \frac{\cos[(\beta L/2)\cos\theta] - \cos(\beta L/2)}{\sin\theta}}$$

ϕ independent



Note: $\beta = \frac{2\pi}{\lambda}$; $\vec{H} = \frac{1}{\eta}\hat{r}\times\vec{E} = \frac{1}{\eta}E_\theta\,\hat{\phi}$

Resonant Antennas: Wires (5)
Patterns



Resonant Antennas: Wires (6)

$\frac{1}{2}\mathrm{Re}\left[\int \vec{E}\times\vec{H}^{*}\cdot d\vec{S}\right]$

Radiated power: $$P = \frac{1}{2\eta}\int_0^{2\pi}\int_0^{\pi}\eta^2\frac{I_m^2}{(4\pi r)^2}\left\{\frac{\cos[(\beta L/2)\cos\theta]-\cos(\beta L/2)}{\sin\theta}\right\}^2 r^2\sin\theta\,d\theta\,d\phi$$

Observation: In general, this is a difficult integral to perform analytically. It is usually done numerically for given βL.

Then: Radiation resistance $\Longrightarrow R_r = \dfrac{2P}{I_m^2}$

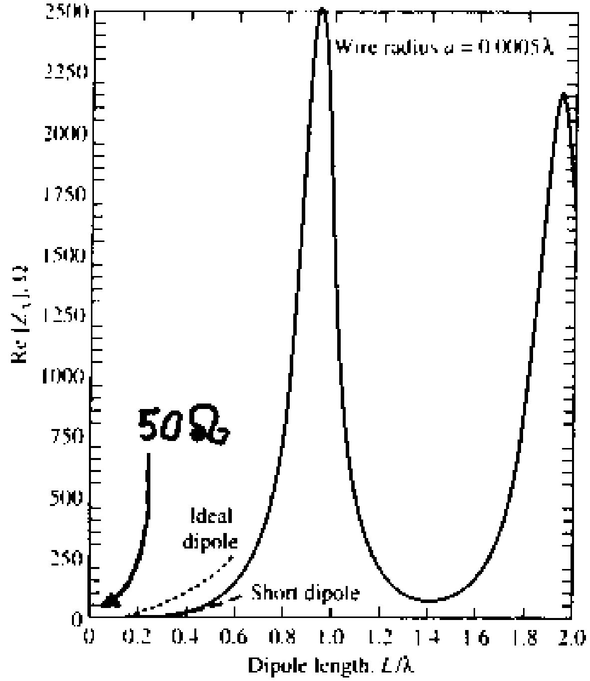
Special case of $\underline{L=\lambda/2}$: $$P = \frac{2.44\eta}{8\pi}I_m^2 \Longrightarrow \boxed{R_r = 73\,\Omega}$$

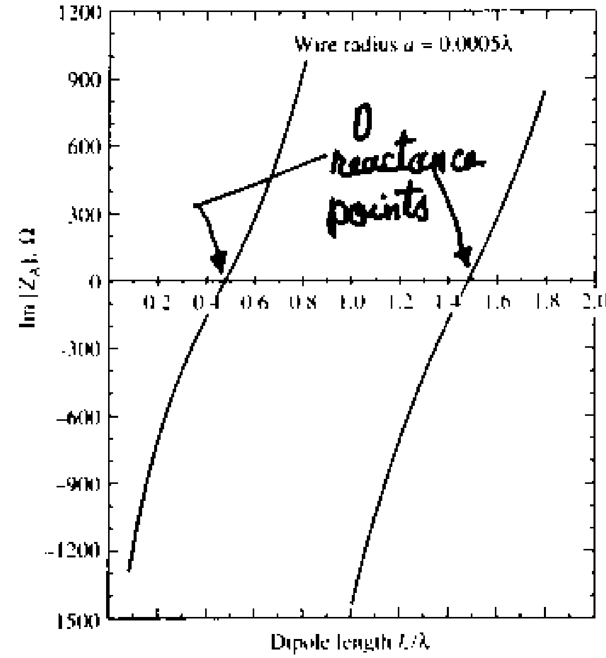
$\eta = \sqrt{\frac{\mu}{\epsilon}} = 120\pi$

Very important result.

Using method of moments: $$\boxed{Z_A = 73 + j\underbrace{42.5}_{\text{inductive reactance}}\,\Omega}$$

Resonant Antennas: Wires (7)
Input Impedance





Ideal dipole and short dipole have very large reactance and therefore hard to match.

Length L	Input Resistance (R_r), Ω
$0 < L < \frac{\lambda}{4}$	$20\pi^2\left(\frac{L}{\lambda}\right)^2$
$\frac{\lambda}{4} < L < \frac{\lambda}{2}$	$24.7\left(\pi\frac{L}{\lambda}\right)^{2.4}$
$\frac{\lambda}{2} < L < 0.637\lambda$	$11.14\left(\pi\frac{L}{\lambda}\right)^{4.17}$

Note: Resonance

Near $\lambda/2$ and $3\lambda/2$, the reactance is 0 and resistance is near 50 Ω.

Resonant Antennas: Wires (8)

The ideal matching situation requires that the reactance of antenna's input impedance is Zero. Based on the method of moments computations, it has been found that depending on the radius of thin dipole antenna, slightly shorter than $\lambda/2$ length gives Zero reactance.

Length for 0 reactance

Length to Diameter Ratio, $L/2a$	Percent Shortening Required	Resonant Length L	Dipole Thickness Class
5000	2	0.49λ	Very thin
50	5	0.475λ	Thin
10	9	0.455λ	Thick

Directivity of $\lambda/2$ dipole

$$\mathcal{D} = \frac{4\pi U_m}{P}$$

$$U_m = \frac{r^2}{2\eta}|E_\theta|^2_{max} = \frac{\eta}{8\pi^2} I_m^2$$

$$P = \frac{2.44\eta}{8\pi} I_m^2$$

$$\boxed{\mathcal{D} = 1.64 = 2.15\,\text{dB}}$$

$10\log 1.64$

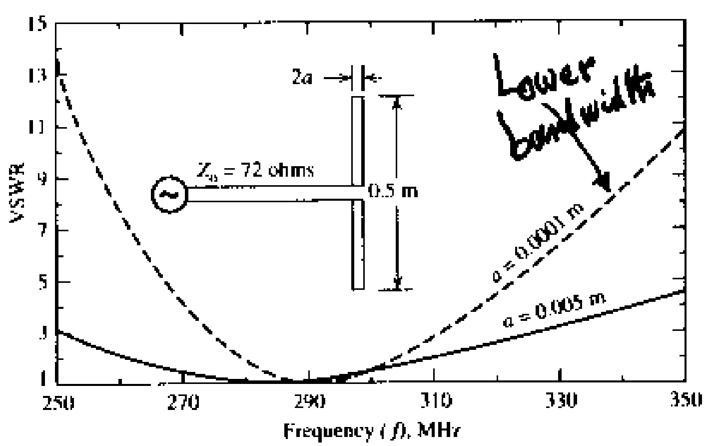
effective aperture

$$\boxed{A_e = \frac{\lambda^2}{4\pi}\mathcal{D} = 0.13\lambda^2}$$

Resonant Antennas: Wires (9)
Bandwidth

Definition: "The range of frequencies within which the performance of the antenna, with respect to some characteristic, conforms to a specified standard".

Observation: Resonant antennas typically have low bandwidth.



Typically VSWR (Voltage standing wave ratio) is used as a measure of the antenna's input impedance to a transmission line.

"read two to one"

Typically VSWR of 2.0:1 is used to define impedance bandwidth.

Example: for VSWR of 2:1

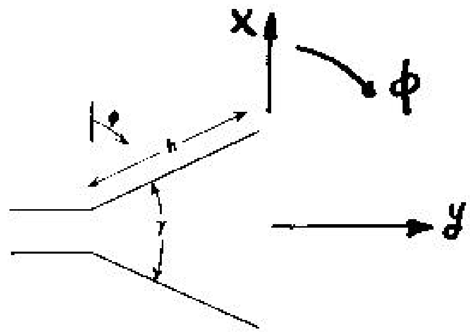
$a = 0.005^m$ $\quad 310 - 262 = 48$ MHz $= 16\%$

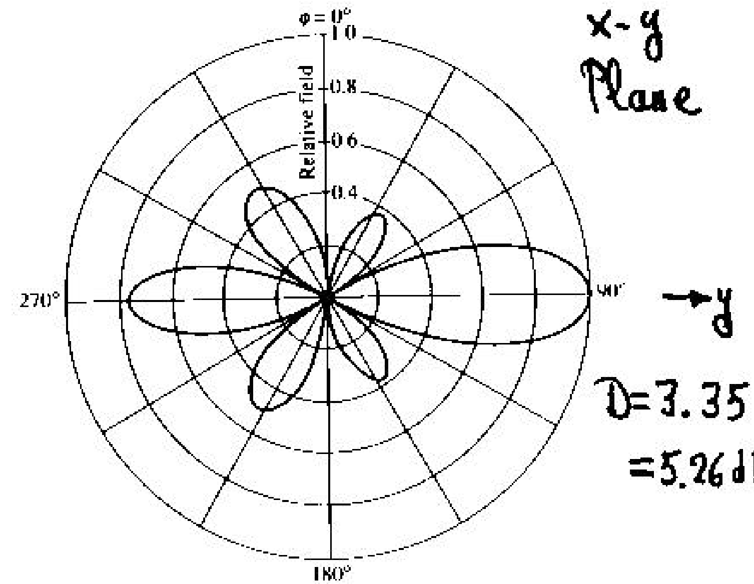
$a = 0.0001^m$ $\quad 304 - 280 = 24$ MHz $= 8\%$

With respect to design freq. of 300 MHz.

Resonant Antennas: Vee dipole

Wire dipole antennas are not always straight. Using MoM, One is able to characterize various antenna topologies.





x-y Plane

$D = 3.35$
$= 5.26\,dB$

Optimum γ angle:

$$\gamma = 152\left(\frac{h}{\lambda}\right)^2 - 388\left(\frac{h}{\lambda}\right) + 324, \quad 0.5 \le \frac{h}{\lambda} < 1.5$$

$$\gamma = 11.5\left(\frac{h}{\lambda}\right)^2 - 70.5\left(\frac{h}{\lambda}\right) + 162, \quad 1.5 \le \frac{h}{\lambda} \le 3.0$$

Directivity:

$$D = 2.95\left(\frac{h}{\lambda}\right) + 1.15$$

$h = 0.75\lambda$; $\gamma = 118.5°$

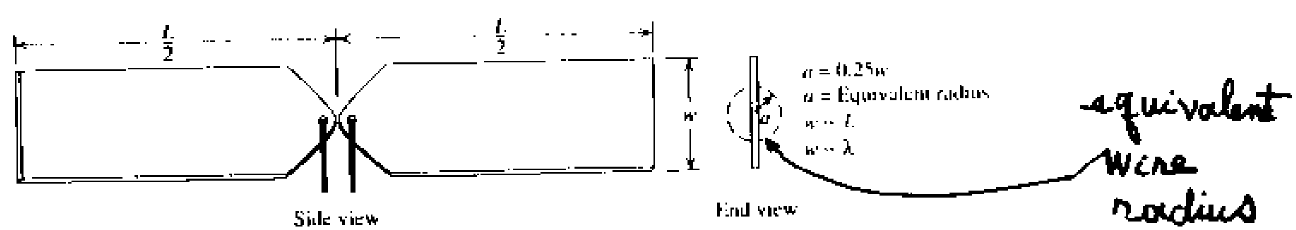
$a = 0.0005\lambda$

$Z_A \approx 106 + j17\ \Omega$

Resonant Antennas: Strip dipole

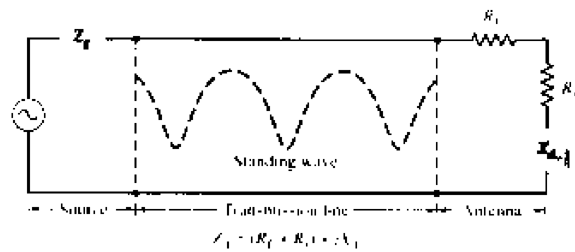
From construction point of view and lower cost of production, occassionally strip dipole is used instead of wire dipole.

For a given wire dipole dimensions, the corresponding strip dipole dimensions are :

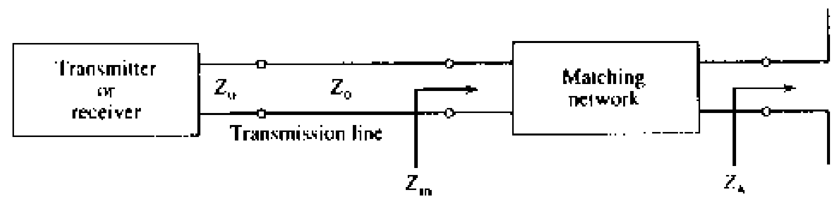


Resonant Antennas: Impedance Matching

- In general antenna's input impedance does not match with the impedance of the transmission.



- In order to reduce the mismatch and prevent losses due to back a forth reflections on the transmissio line, matching network are used.



VSWR	Percent Reflected Power $= \lvert\Gamma\rvert^2 \times 100 = \left(\frac{\text{VSWR} - 1}{\text{VSWR} + 1}\right)^2 \times 100$	Percent Transmited Power $= q \times 100 = (1 - \lvert\Gamma\rvert^2) \times 100$
1.0	0.0	100.0
1.1	0.2	99.8
1.2	0.8	99.2
1.5	4.0	96.0
2.0	11.1	88.9
3.0	25.0	75.0
4.0	36.0	64.0
5.0	44.4	55.6
5.83	50.0	50.0
10.0	66.9	33.1

means 89% is transmitted

- Matching network takes many different design depending on The antenna configuration, transmission line, bandwidth objectives, etc.

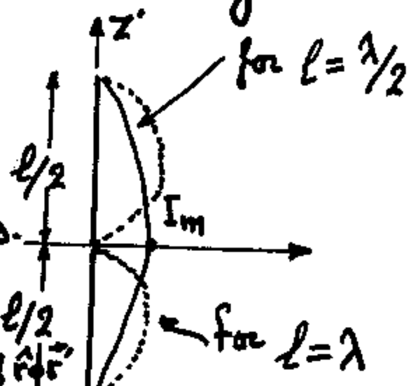
Finite Length Dipole (1)

Current density : $\vec{J}(\vec{r}\,') = \underbrace{I(z')}_{A}\,\underbrace{\delta(x')}_{\frac{1}{m}}\,\underbrace{\delta(y')}_{\frac{1}{m}}\,\hat{z}$ $\quad A/m^2$ $\Longrightarrow$ dipole along z axis

A good approximation for the current : $I(z') = I_m \sin\left[\beta\left(\frac{\ell}{2} - |z'|\right)\right]$ $\Longrightarrow$ dipole length

Note : • Current goes to zero at the ends.

• For different length dipoles, current distribution takes various distributions.



For far field computation:

$$\vec{A} = \mu \frac{e^{-j\beta r}}{4\pi r}\int_V \vec{J}(r')\, e^{j\beta \hat{r}\cdot\vec{r}\,'}\, dv' = \mu \frac{e^{-j\beta r}}{4\pi r}\int_{-\ell/2}^{\ell/2} I(z')\, e^{j\beta \hat{r}\cdot\vec{r}\,'}\, dz'$$

Finite Length Dipole (2)

Among various length dipoles, half-wave length ($\lambda/2$) dipoles are the most popular ones. This is because of their "good" input impedance.

$\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$ $\qquad$ $z'\hat{z}$

Recall for $\hat{z}$ oriented linear current ($\hat{r}\cdot\vec{r}' = z'\cos\theta$)

E-field: $$E_\theta = -j\omega A_\theta = j\omega \sin\theta\, A_z = j\omega\mu \sin\theta \frac{e^{-j\beta r}}{4\pi r} \underbrace{\int I(z') e^{j\beta z'\cos\theta} dz'}_{f_{un}}$$

We need to evaluate this integral

For $\ell = \lambda/2$

$$f_{un} = I_m \int_{-\lambda/4}^{0} \sin\left(\frac{\pi}{2} + \beta z'\right) e^{j\beta z'\cos\theta} dz'$$

$$+ I_m \int_{0}^{\lambda/4} \sin\left(\frac{\pi}{2} - \beta z'\right) e^{j\beta z'\cos\theta} dz'$$

Finite Length Dipole (3)

Integral Identity: $\int \sin(a+bx)\, e^{cx}\, dx = \dfrac{e^{cx}}{b^2+c^2}\left[c\sin(a+bx) - b\cos(a+bx)\right]$

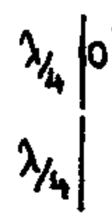
thereafter some manipulations: $f_{un} = \dfrac{I_m}{\beta \sin^2\theta}\, 2\cos\left(\frac{\pi}{2}\cos\theta\right)$

Finally: $$E_\theta = j\omega\mu \frac{2I_m}{\beta} \frac{e^{-jkr}}{4\pi r} \sin\theta \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

radiation pattern

element pattern. Each differential current element radiates $\sin\theta$

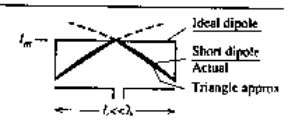
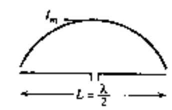
pattern factor



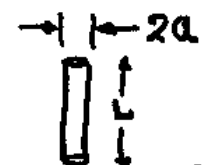
Pattern: $F(\theta) = \sin\theta \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$

Note: at $\theta = \pi/2$; $F(\pi/2) = 1$

$\lambda/2$ Dipole vs. ideal & short dipoles

Dipole Type	Length	Current	Pattern	HP	D	D (dB)	R_r (Ω)	R_{ohmic} (Ω)	Current Distribution
Ideal	$L \ll \lambda$	Uniform	$\sin\theta$	90°	1.5	1.76	$80\pi^2\left(\frac{L}{\lambda}\right)^2$	$\frac{R_s}{2\pi a}L$	
Short	$L \ll \lambda$	Triangle	$\sin\theta$	90°	1.5	1.76	$20\pi^2\left(\frac{L}{\lambda}\right)^2$	$\frac{R_s}{2\pi a}\frac{L}{3}$	
Half-wave	$L = 0.5\lambda$	Sinusoid	$\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$	78°	1.64	2.15	~70	$\frac{R_s}{2\pi a}\frac{\lambda}{4}$	

Note: For an antenna of length L that carries an axial uniform current

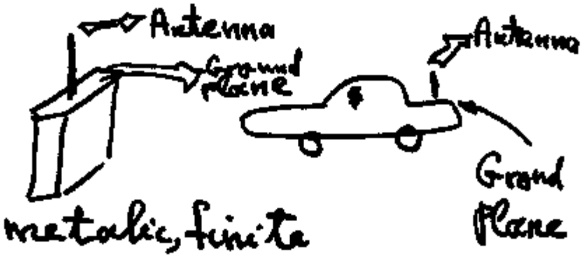


$$R_{ohmic} \approx \frac{L}{2\pi a} R_s$$ surface resistance

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} \; ; \; \omega = 2\pi f$$ frequency

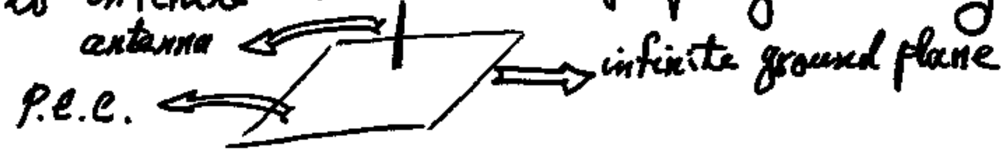
Antennas above a Perfect Ground Plane
Image theory & Monopole

- In many practical situations, antennas are mounted on ground planes.



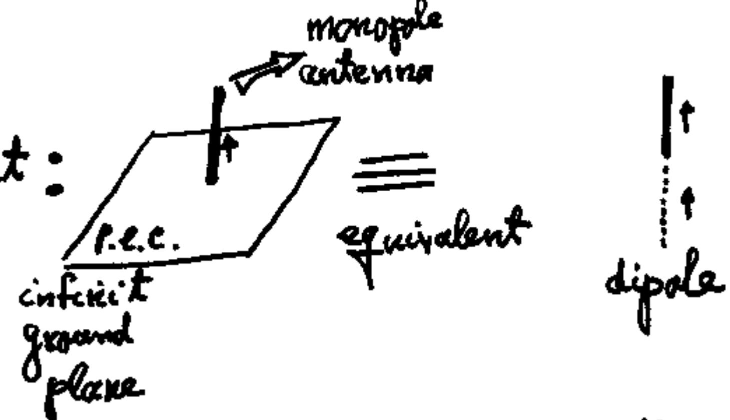
- In practice, ground planes are metalic, finite size and may not be planar.

- A useful approximation that allows the application of image theory is to assume that the ground plane is "infinite" in extend and perfectly conducting (P.E.C.).

- 

Antennas above a Perfect Ground Plane
Image theory & Monopole

- Image theory allows one to create equivalent known problem:



- In the upper hemisphere, monopole creates exactly the same electromagnetic fields as the dipole.

ideal Monopole

$$D_{mono} = \frac{4\pi}{\Omega_{A,mono}} = \frac{4\pi}{\frac{1}{2}\Omega_{A,dipole}} = 2D_{dipole}$$

Directivity of a monopole is twice of dipole.

$$R_{r,mono} = \frac{1}{2}R_{r,dipole} = 40\pi^2\left(\frac{h}{\lambda}\right)^2$$

h: length of monopole

Small Loop Antennas

Small loop antennas:

perimeter $<< \lambda$

uniform time varying current

It turns out that the radiation fields of small loops are independent of the shape of the loop and depend only on the area of the loop.

Vector potential

current along the loop

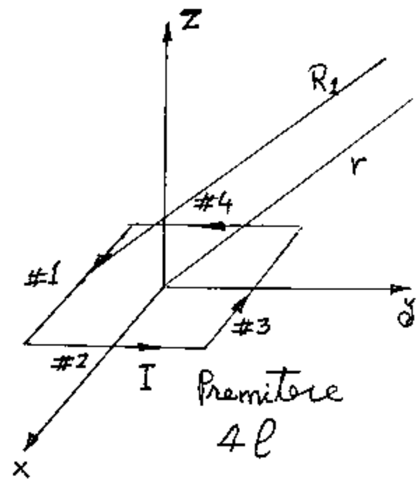
$$\vec{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \iiint_{V'} \vec{J}\, e^{j\beta \hat{r}\cdot\vec{r}\,'}\, dv' = \mu \frac{e^{-j\beta r}}{4\pi r} \int_C \vec{I}\, e^{j\beta \hat{r}\cdot\vec{r}\,'}\, dc'$$

along the loop

Note: It appears to be easier to construct $\vec{A}$ for the square loop.

Small Loop Antennas

Square Loop



Small square loop can be thought of as combination of "4" ideal dipole.

$$\vec{A}_{\substack{small\\square}} = \hat{x} A_1 + \hat{y} A_2 + \hat{x} A_3 + \hat{y} A_4$$

or

$$\begin{cases} A_x = \dfrac{\mu I \ell}{4\pi}\left(\dfrac{e^{-j\beta R_1}}{R_1} - \dfrac{e^{-j\beta R_3}}{R_3}\right) \\[2ex] A_y = \dfrac{\mu I \ell}{4\pi}\left(\dfrac{e^{-j\beta R_2}}{R_2} - \dfrac{e^{-j\beta R_4}}{R_4}\right) \end{cases}$$

For phase term

$$\begin{cases} R_1 = r + \frac{\ell}{2}\sin\theta\sin\phi \\ R_3 = r - \frac{\ell}{2}\sin\theta\sin\phi \end{cases}$$

$$\begin{cases} R_2 = r - \frac{\ell}{2}\sin\theta\cos\phi \\ R_4 = r + \frac{\ell}{2}\sin\theta\cos\phi \end{cases}$$

For far field approximation: for amplitude term : $R_1 \approx R_2 \approx R_3 \approx R_4 \approx R$

Small Loop Antennas

Square loop

Then: $$A_x = \frac{\mu I \ell e^{-j\beta r}}{4\pi r}\left[e^{-j\beta(\ell/2)\sin\theta\sin\phi} - e^{+j\beta(\ell/2)\sin\theta\sin\phi}\right]$$

$\ell \ll \lambda$

for small $\beta\ell = 2\pi\frac{\ell}{\lambda}$

recall: $$\sin\alpha = \frac{1}{2j}\left(e^{j\alpha} - e^{-j\alpha}\right)$$

$\frac{\beta\ell}{2}\sin\theta\sin\phi$

Finally: $$A_x = -2j\,\frac{\mu I \ell e^{-j\beta r}}{4\pi r}\sin\left(\frac{\beta\ell}{2}\sin\theta\sin\phi\right)$$

A_y: similarly

or: $$\vec{A} = A_x\hat{x} + A_y\hat{y} = j\beta\ell^2\,\frac{\mu I e^{-j\beta r}}{4\pi r}\sin\theta\,\underbrace{(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})}_{\hat{\phi}}$$

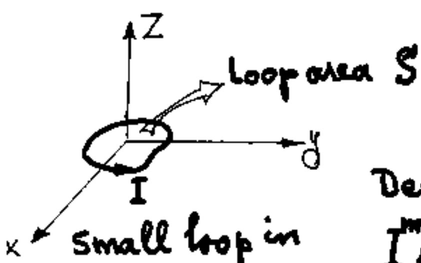
$\vec{E}$ & $\vec{H}$ Fields: $\vec{E} = -j\omega\vec{A} \Rightarrow$

$$\boxed{\begin{aligned}\vec{E} &= \eta\beta^2 S\,\frac{I e^{-j\beta r}}{4\pi r}\sin\theta\,\hat{\phi}\\ \vec{H} &= \frac{1}{\eta}\hat{r}\times\vec{E} = -\beta^2 S\,\frac{I e^{-j\beta r}}{4\pi r}\sin\theta\,\hat{\theta}\end{aligned}}$$

Note: $S = \ell^2$ (area of loop)

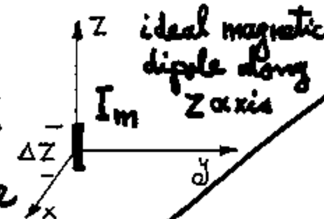
$\omega\mu\beta = \eta\beta^2$

Small loop vs. Ideal dipole



small loop in x-y plane

$\equiv$



Define

$$I^m \Delta Z = j\omega\mu I S$$

$$\vec{E} = -j\omega\mu\, I_m \Delta Z\, \frac{e^{-j\beta r}}{4\pi r} \sin\theta\, \hat{\phi}$$

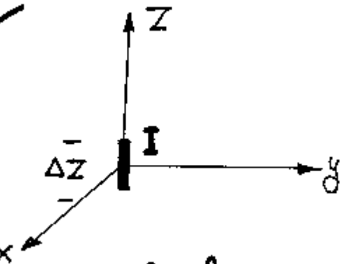
$$\vec{H} = j\omega\sqrt{\mu\epsilon}\, I_m \Delta Z\, \frac{e^{-j\beta r}}{4\pi r} \sin\theta\, \hat{\theta}$$

$$HP = 90^\circ$$

$$\Omega_A = \frac{8\pi}{3}$$

$$D = \frac{3}{2} \text{ or } 1.75 \text{ dB}$$

$$R_r = 80\pi^2 \left(\frac{S}{\lambda^2}\right)^2 \text{ ohm}$$



ideal dipole along Z axis

$$\vec{E} = j\omega\mu\, I \Delta Z\, \frac{e^{-j\beta r}}{4\pi r} \sin\theta\, \hat{\theta}$$

$$\vec{H} = j\omega\sqrt{\mu\epsilon}\, I \Delta Z\, \frac{e^{-j\beta r}}{4\pi r} \sin\theta\, \hat{\phi}$$

$$HP = 90^\circ$$

$$\Omega_A = \frac{8\pi}{3}$$

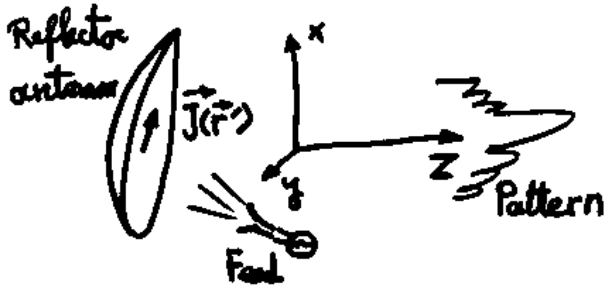
$$D = \frac{3}{2} \text{ or } 1.75 \text{ dB}$$

$$R_r = 80\pi^2 \left(\frac{\Delta Z}{\lambda}\right)^2 \text{ ohm}$$

Antenna Arrays

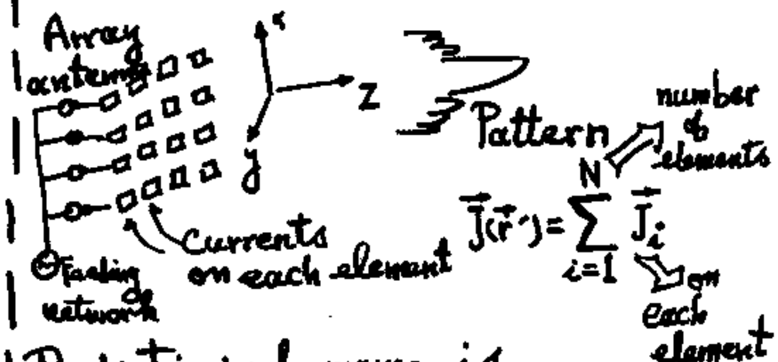
Definition: When two or more antennas are used together, the combination is called "antenna array", "array antenna" or simply an "array".

Continuous currents



Radiation performance is controled by reflector shape, feed design and illumination.

Discrete currents



Radiation performance is controled by element locations, element design and complex excitation coefficients.

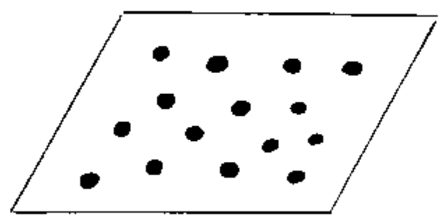
Type of Arrays

Depending on the physical location of the discrete elements, one may classify the array as:

Linear array:

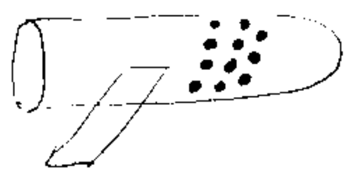
elements are positioned along a straight line.

Planar array:



elements are positioned on a plane.

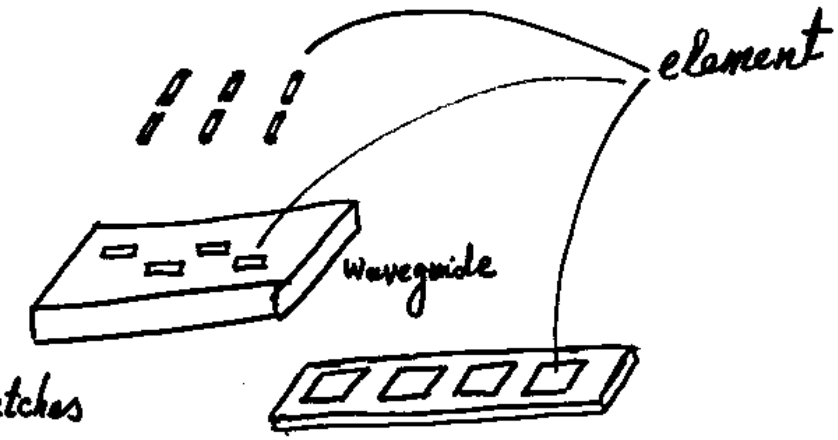
Conformal array:



elements are positioned on a curved surface.

Array Elements

- Generally speaking, any antenna can be used as an array element. In practice, however, the following elements are mostly used
  - dipoles
  - slots
  - Microstrip patches
  - etc.

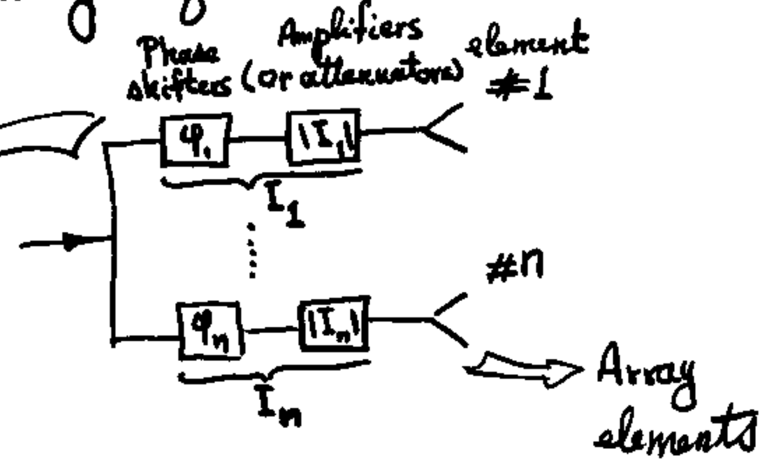


Complex Excitation Coefficients

- Due to the discrete nature of array elements, the "amplitue and phase" of each element can be individually controled. → so called "Excitation coefficients"

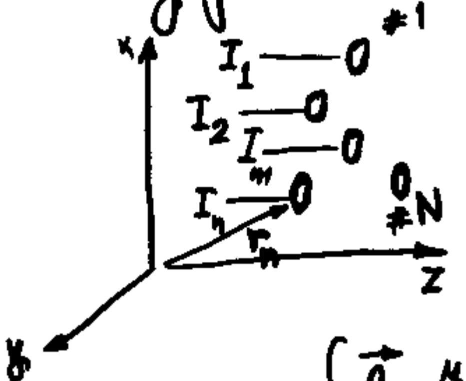
- There are many ways to achieve this.

this is called Beam forming network or Feed network



What is the BIG question for antenna arrays?

Knowing the location of elements, the type of elements and their complex excitation coefficients, find antenna array pattern.



the key question is how to express $\vec{J}$?

Far field:

$$\begin{cases} \vec{A} = \mu \dfrac{e^{-j\beta r}}{4\pi r} \displaystyle\int \vec{J}(\vec{r}\,')\, e^{j\beta \hat{r}\cdot\vec{r}\,'}\, dv' \\ \vec{E} = -j\omega (A_\theta \hat{\theta} + A_\phi \hat{\phi}) \\ \vec{H} = \dfrac{1}{\eta}\, \hat{r} \times \vec{E} \end{cases}$$

$$\vec{J}(\vec{r}\,') = \sum_{i=1}^{N} \vec{J}_i$$

$$\sum_{i=1}^{N} \int_{\text{on each element}} \vec{J}_i\, e^{j\beta \hat{r}\cdot\vec{r}\,'}\, dv'$$

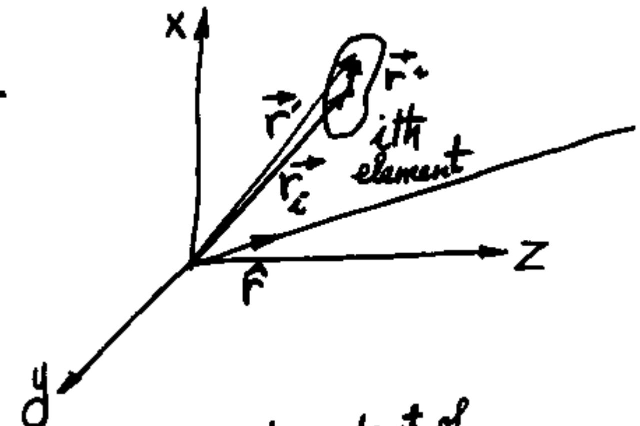
Array Pattern Construction (1)

$\vec{r}_i$: i^{th} element position vector

$\vec{r}\,'$: source variable vector

$\vec{r}_i\,''$: source variable vector local to the element

$\hat{r}$: Unit vector for position observation



independent of integration variable

depends on integration variable

Note: $\vec{r}\,' = \vec{r}_i + \vec{r}\,'' \Rightarrow \hat{r}\cdot\vec{r}\,' = \hat{r}\cdot\vec{r}_i + \hat{r}\cdot\vec{r}_i\,''$

Now: $$\boxed{\sum_{i=1}^{N} \int_{\text{on each element}} \vec{J}_i\, e^{j\beta \hat{r}\cdot\vec{r}\,'}\, dv' = \sum_{i=1}^{N} e^{j\beta \hat{r}\cdot\vec{r}_i} \int_{\text{on } i^{th} \text{ element}} \vec{J}_i(\vec{r}\,'')\, e^{j\beta \hat{r}\cdot\vec{r}\,''}\, dv''}$$

Array Pattern Construction (2)

Define normalized element current: $\vec{J}_i(\vec{r}'') = I_i \; \vec{J}_{i\,nor}(\vec{r}'')$

normalized current

Constant complex excitation coefficient $(|I_i| \angle \text{Phase})$

then:

Array vector potential: $$\vec{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \sum_{i=1}^{N} I_i \, e^{j\beta \hat{r}\cdot\vec{r}_i} \int_{i\text{th element}} \vec{J}_{i\,nor} \, e^{j\beta \hat{r}\cdot\vec{r}''} \, dv''$$

ith element vector potential $\vec{A}_i$

Or: $$\boxed{\vec{E}_{\substack{\text{array}\\\text{antenna}}}(\vec{r}) = -j\omega\mu \frac{e^{-j\beta r}}{4\pi r} \sum_{i=1}^{N} I_i \, e^{j\beta \hat{r}\cdot\vec{r}_i} \underbrace{\left(A_{i\theta}\hat{\theta} + A_{i\varphi}\hat{\varphi}\right)}_{\text{related to the element pattern}}}$$

This is a very general result.

related to the element pattern

Array Pattern Construction (3)
Identical elements

In most applications array elements are identical in their design, shape and orientation. Therefore,

$$A_{i\theta}\hat{\theta} + A_{i\phi}\hat{\phi} = \underbrace{A_{ele\,\theta}\hat{\theta} + A_{ele\,\phi}\hat{\phi}}_{\text{the same for all elements}}$$

then:

$$\underset{\substack{\text{array}\\\text{antenna}}}{\vec{E}}(\vec{r}) = -j\omega\mu\frac{e^{-j\beta r}}{4\pi r}\left(A_{ele\,\theta}\hat{\theta} + A_{ele\,\phi}\hat{\phi}\right)\sum_{i=1}^{N} I_i\, e^{j\beta\hat{r}\cdot\vec{r}_i}$$

array pattern = element pattern × array factor

Array Factor : This is a scalar quantity

Array factor plays paramont role in controling the radiation characteristics of array antennas.

Array Factor : $$\boxed{AF = \sum_{i=1}^{N} I_i \, e^{j\beta \hat{r}\cdot\vec{r}_i}}$$

observation direction

ith element position

ith element complex excitation coefficient

propagation constant $\beta = \frac{2\pi}{\lambda}$

λ Wavelength

$$I_i = |I_i| e^{j\varphi_i}$$

phase

magnitue

Wavelength $\lambda = \frac{c}{f}$

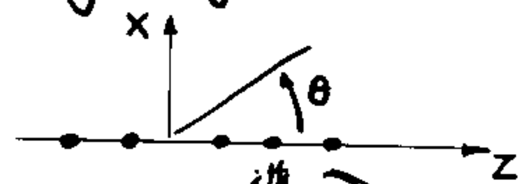
c speed of light

f frequency

$$\begin{cases} \hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \\ \vec{r}_i = x_i\,\hat{x} + y_i\,\hat{y} + z_i\,\hat{z} \end{cases}$$

$$\boxed{\hat{r}\cdot\hat{r}_i = x_i\sin\theta\cos\phi + y_i\sin\theta\sin\phi + z_i\cos\theta}$$

Array Factor: Linear Array along Z axis



i^{th} element → location Z_i & excitation coefficient I_i

Note: $\vec{r}_i = z_i \hat{z}$

therefore: $\hat{r} \cdot \vec{r}_i = z_i \cos\theta$

then Array Factor becomes:

$$\boxed{AF = \sum_{i=1}^{N} I_i \, e^{j\beta z_i \cos\theta}}$$

Linear array with identical excitation coefficients: $I_i = I = 1$ → for the sake of simplicity

then

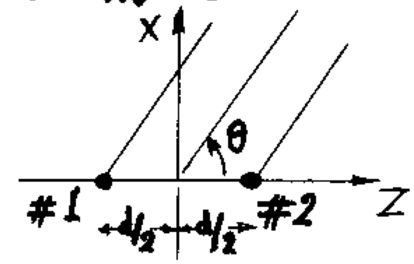
$$\boxed{AF = \sum_{i=1}^{N} e^{j\beta z_i \cos\theta}}$$

Note: For linear array long Z axis, AF is independent of ϕ angle.

Example: Array Factor for two elements along z axis

Element locations $\vec{r}_1 = -\frac{d}{2}\hat{z}$

$\vec{r}_2 = +\frac{d}{2}\hat{z}$



Array Factor for two elements: $AF = I_1 e^{-j\beta(d/2)\cos\theta} + I_2 e^{j\beta(d/2)\cos\theta}$

Special case $I_1 = I_2 = 1$ $\quad \boxed{AF = 2\cos(\beta \frac{d}{2}\cos\theta)}$: Elements in phase

Special case
$I_1 = -1 = 1.\, e^{j180°}$
$I_2 = 1. = 1.\, e^{j0°}$

$\boxed{AF = 2j\sin(\beta \frac{d}{2}\cos\theta)}$: Elements 180° out of phase

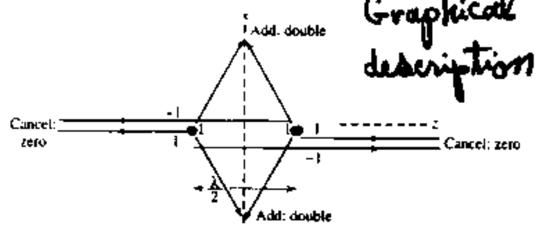
Example: Two elements $\lambda/2$ apart & same amplitude

In phase

$$I_1 = I_2 = 1 \; ; \; d = \lambda/2$$

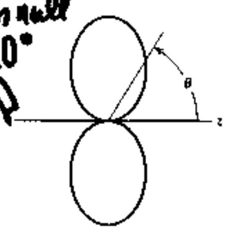
$$AF = 2\cos\left(\frac{\pi}{2}\cos\theta\right)$$

Graphical description

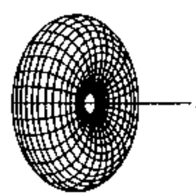


(a) Inspection method.

Pattern null
$\theta = 0°$



(b) Polar plot of the array factor $f(\theta) = \cos[(\pi/2)\cos\theta]$.



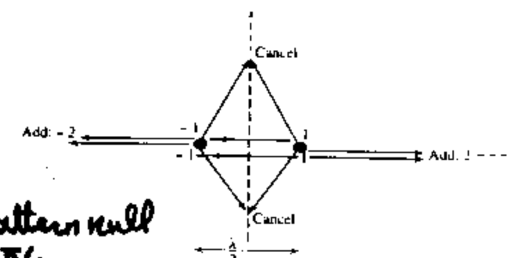
(c) 3D polar pattern.

Normalized AF

out of phase

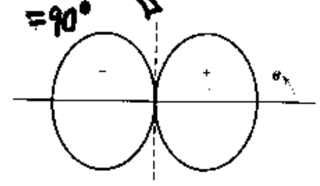
$$I_1 = -1, \; I_2 = 1 \; ; \; d = \lambda/2$$

$$AF = 2j\sin\left(\frac{\pi}{2}\cos\theta\right)$$

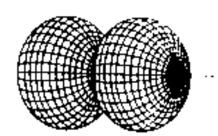


(a) Inspection method.

Pattern null
$\theta = \pi/2$
$= 90°$



(b) Polar plot of the array factor magnitude $|f(\theta)| = |\sin[(\pi/2)\cos\theta]|$.



(c) 3-D polar pattern.

Normalized AF

Example: Two elements $\lambda/4$ apart, same amplitude & 90° out of phase

$$I_1 = 1, \quad I_2 = 1.0\, e^{-j\pi/2}; \quad d = \lambda/4$$

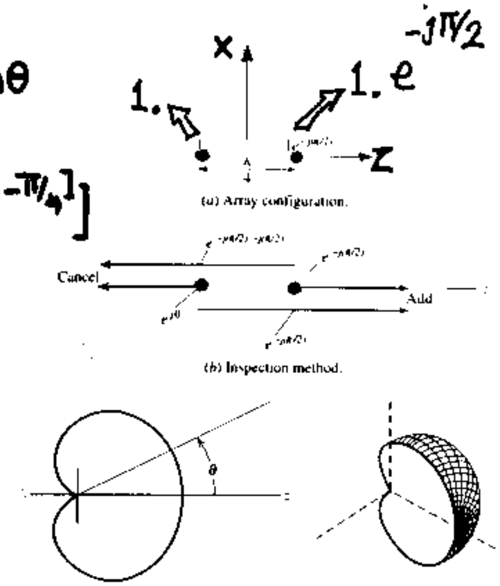
$$AF = 1\, e^{-j\beta(d/2)\cos\theta} + 1\, e^{-j\pi/2}\, e^{j\beta(d/2)\cos\theta}$$

$$= e^{-j\pi/4}\left[e^{-j[\beta(d/2)\cos\theta - \pi/4]} + e^{j[\beta(d/2)\cos\theta - \pi/4]} \right]$$

$$= e^{-j\pi/4}\, 2\cos\left(\frac{\beta d}{2}\cos\theta - \frac{\pi}{4}\right)$$

For $d = \lambda/4$

$$AF = 2\, e^{-j\pi/4} \underbrace{\cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]}_{\text{Normalized AF}}$$



(a) Array configuration.

(b) Inspection method.

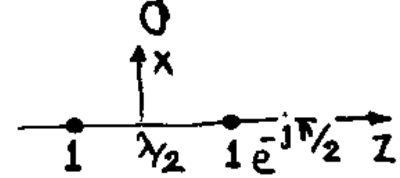
(c) Polar plot of the array factor $f(\theta) = \cos[(\pi/4)(\cos\theta - 1)]$.

(d) 3-D polar pattern

Observation:

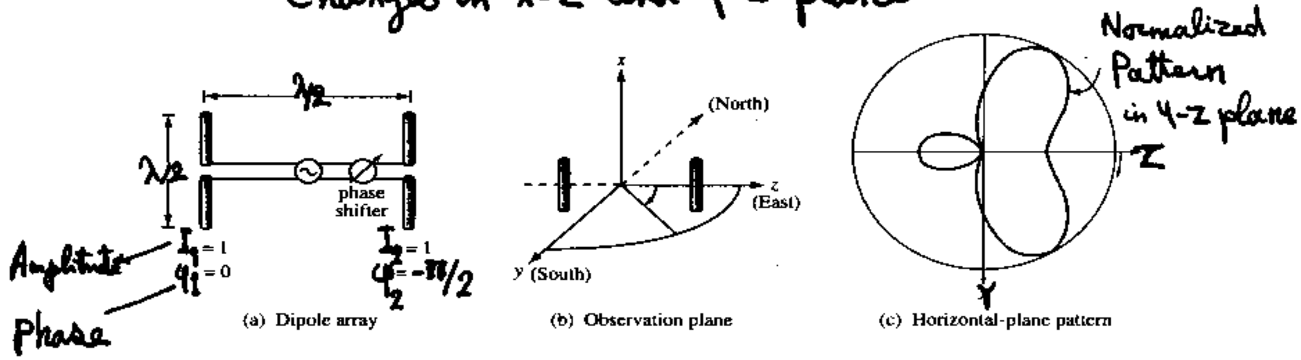
Clearly these patterns are controled by excitation coefficients!

Example: Two-element $\lambda/2$-dipole Array



Note: • In this case AF is ϕ Symmetric.

• The antenna pattern is not ϕ symmetric, due to the fact the element pattern changes in x-z and y-z plane.

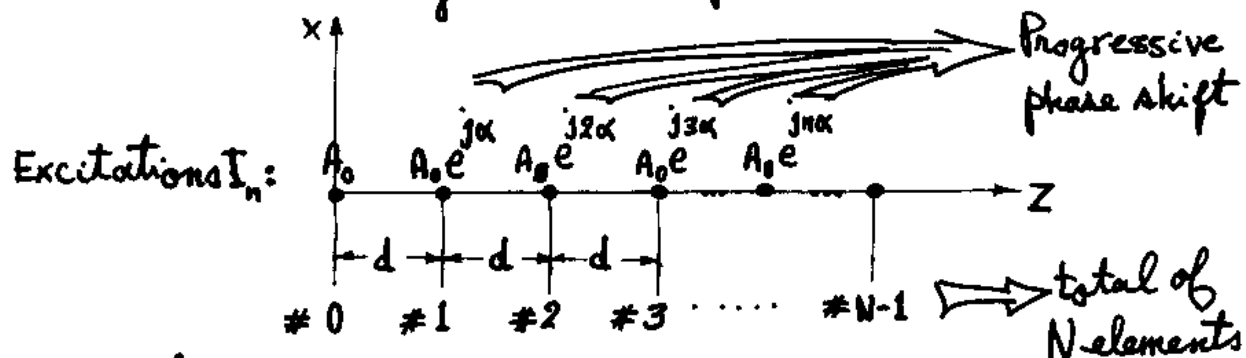


(a) Dipole array (b) Observation plane (c) Horizontal-plane pattern

Uniform Arrays (1)

Definition: An array of identical elements all of identical magnitude and each with a progressive phase is referred to as a "uniform array".

Importance: This is an important class of array when one desires to change the element phase to scan the beam.



Note: Element numbering is the user's choice. One could also use #1, ..., #N.

Uniform Arrays (2)

Array Factor: Recall

$$AF = \sum_{n=0}^{N-1} I_n \, e^{j\beta z_n \cos\theta}$$

$\beta \rightarrow 2\pi/\lambda$

$z_n \rightarrow$ location of n^{th} element

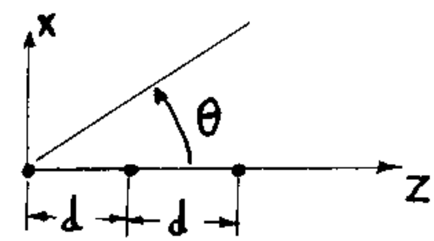
$I_n \rightarrow$ Excitation coefficient of n^{th} element

For the uniform array: $I_n = A_0 \, e^{jn\alpha}$; $z_n = nd$

then:

$$AF = A_0 \sum_{n=1}^{N-1} e^{jn\alpha} \, e^{j\beta nd\cos\theta} = A_0 \sum_{n=1}^{N-1} e^{jn(\beta d\cos\theta + \alpha)}$$

Let: $\underline{\psi = \beta d \cos\theta + \alpha}$



Finally:

$$\boxed{AF = A_0 \sum_{n=0}^{N-1} e^{jn\psi}}$$

$\Rightarrow$ Can this be simplified? Yes

Uniform Arrays (3)

It has been shown: $AF = A_0 \sum_{n=0}^{N-1} e^{jn\psi} = A_0 (1 + e^{j\psi} + \cdots + e^{j(N-1)\psi})$

Multiply each side by $e^{j\psi}$: $e^{j\psi} AF = A_0 (e^{j\psi} + e^{j2\psi} + \cdots + e^{jN\psi})$

subtract these two

then: $(1 - e^{j\psi}) AF = A_0 (1 - e^{jN\psi}) \Rightarrow \boxed{AF = A_0 \dfrac{1 - e^{jN\psi}}{1 - e^{j\psi}}}$

More simplification: $AF = A_0 \dfrac{e^{jN\psi/2}}{e^{j\psi/2}} \dfrac{e^{jN\psi/2} - e^{-jN\psi/2}}{e^{j\psi/2} - e^{-j\psi/2}}$

this is a sine $2j \sin N\psi/2$

therefore: $\boxed{AF = A_0 e^{j(N-1)\psi/2} \dfrac{\sin(N\psi/2)}{\sin(\psi/2)}}$

Note: This is not normalized at $\psi = 0$

Uniform Arrays (4)

Aside: What is $\left.\frac{\sin(ax)}{\sin(x)}\right|_{x=0}$? $\left.\frac{ax-\frac{1}{3!}(ax)^3+\cdots}{x-\frac{1}{3!}x^3+\cdots}\right|_{x\to 0} = a$

or $\boxed{\left.\frac{\sin(ax)}{ax}\right|_{x=0} = 1}$

Normalized AF_n : $\boxed{AF_n = \frac{\sin(N\psi/2)}{N\sin(\psi/2)}}$

Note: $\psi \to 0$ (or $2p\pi$) — p: an integer

$AF_n = 1$ $\Rightarrow$ normalized

Recall : $\psi = \beta d\cos\theta + \alpha$

ψ: depends on many factors

β: depends on frequency

α: progressive phase shift

θ: angle of observation

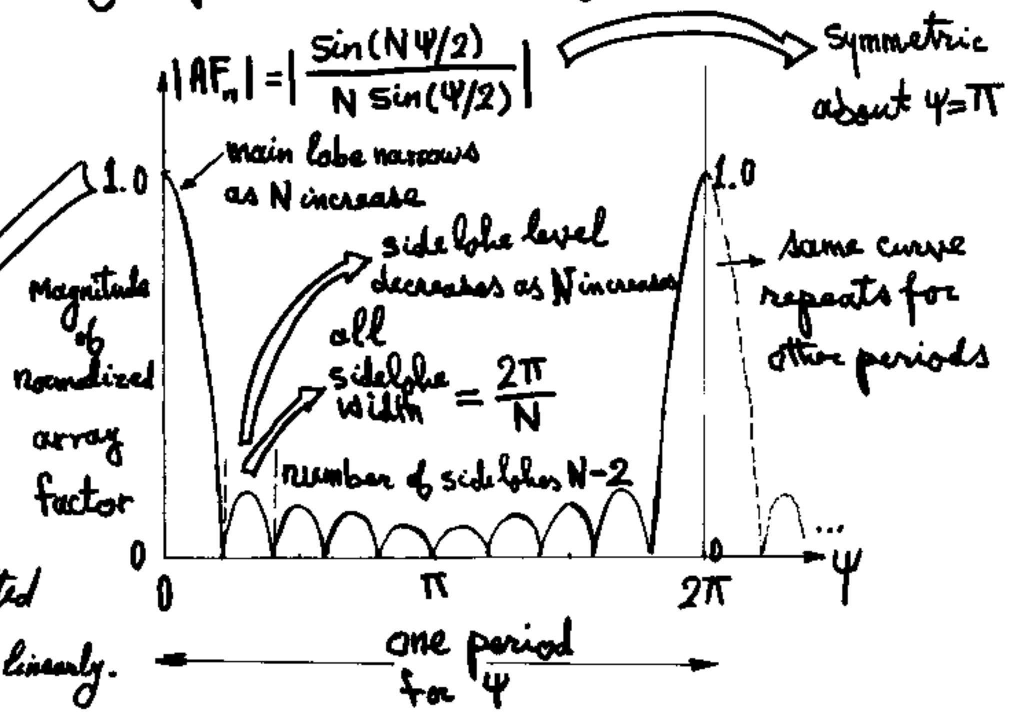
d: element spacing

Universal Pattern for Uniform array : $\boxed{AF_n : \text{Only depends on } \psi \text{ and } N}$

Uniform Arrays (5): Universal Pattern

Key feature: As a function of ψ, Universal pattern only depends on the number of elements N.

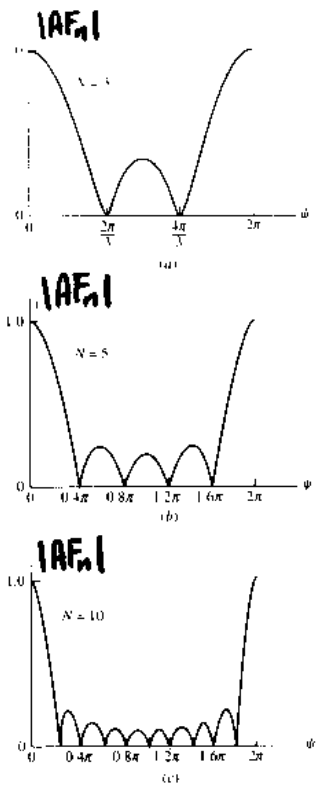
Graphical features:



This is in a linear scale.
Can also be plotted in "log" scale linearly.

Uniform Arrays (6): Universal Pattern

Examples:



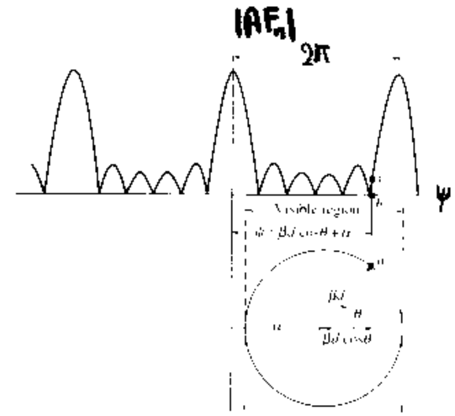
$$\frac{\sin\left(\frac{3\psi}{2}\right)}{3\sin\frac{\psi}{2}}$$

$$\frac{\sin\frac{5\psi}{2}}{5\sin\frac{\psi}{2}}$$

$$\frac{\sin\frac{10\psi}{2}}{10\sin\frac{\psi}{2}}$$

RectiLinear (amplitude or log) to Polar (amplitude or log) graphical Construction (1)

Universal patterns are a generalized way to present the normalized array factors of Uniform arrays. It is, however, desirable to use the universal pattern and generate the polar pattern versus angle θ.



Steps

1. Plot $|AF_n|$ in rectilinear coordinates (in amplitude & log)
2. Draw a circle with radius "βd" and with its center at $\psi = \alpha$
3. Draw vertical lines to intersect the circle.
4. From the center, draw radial lines to these intersecting points and identify θ.
5. Along these radial lines mark off corresponding magnitudes.

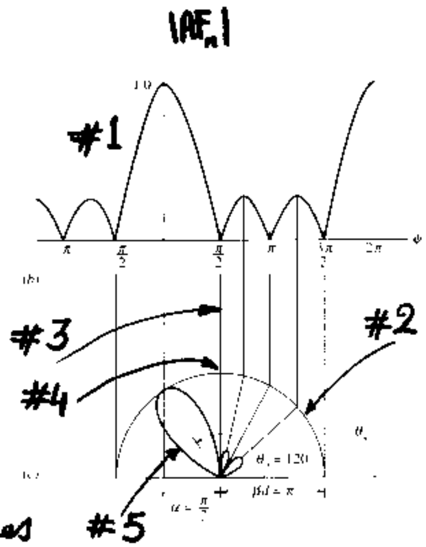
Rectilinear to Polar graphical Construction (2)

Example: 4-element linear array

uniform array configuration: d — $\lambda/2$ — $\lambda/2$ — $\lambda/2$

Excitation: $1 \quad 1e^{j\pi/2} \quad 1e^{j\pi} \quad 1e^{j3\pi/2}$ → Progressive phase shift $\alpha = \pi/2$

Note: $\boxed{\alpha = \pi/2 \;;\; \beta d = \frac{2\pi}{\lambda} d = \pi \qquad N = 4}$



Steps:

1. $|AF_n| = \left| \dfrac{\sin 2\psi}{4 \sin \frac{\psi}{2}} \right|$
2. Circle radius "π" centered at "$\pi/2$".
3. Vertical lines
4. Intersection points on the circle
5. Corresponding magnitudes along radial lines

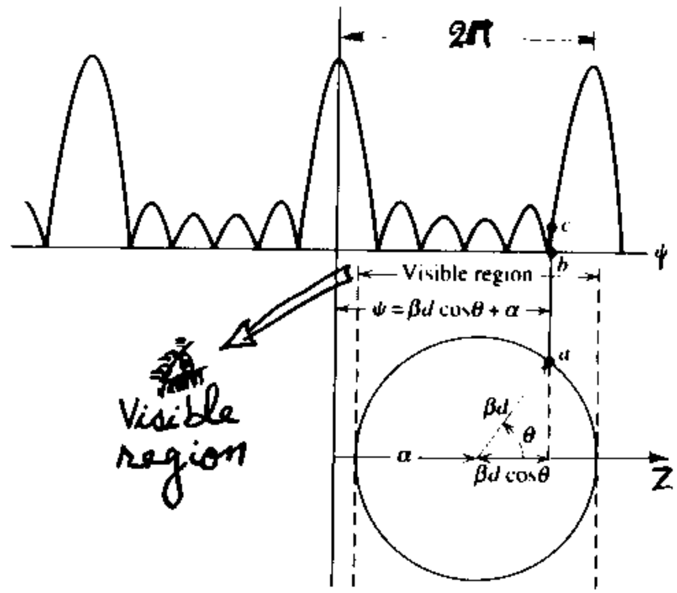
Visible and Invisible Regions

Properties of $AF(\psi)$

- Array factor is periodic in variable ψ with 2π period.

$$AF(\psi) = AF(\psi + 2\pi)$$

- The portion of the universal pattern which projects onto the circle with radius "$\beta d \cos\theta$" belongs to the visible range.
- The rest belongs to the invisible range.



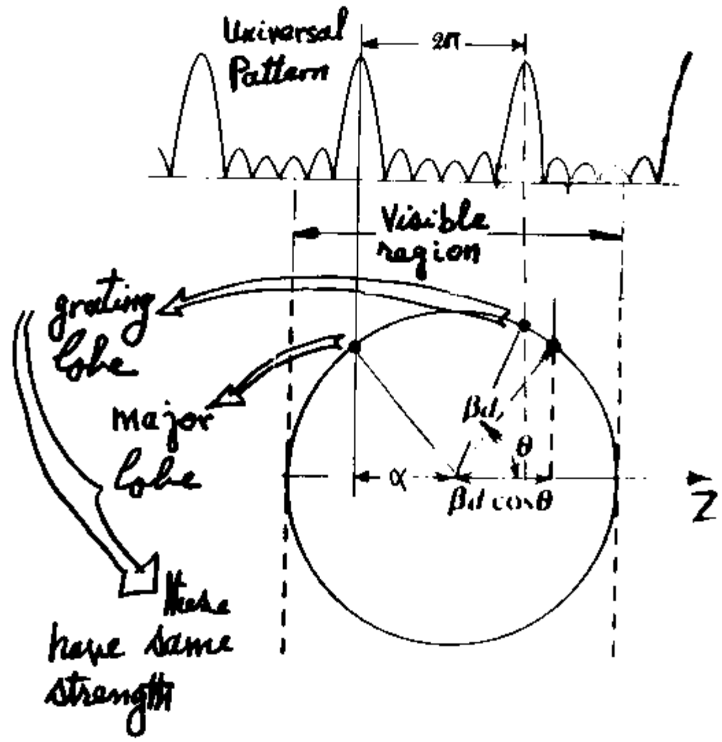
- For linear array along z axis visible region is

$$0 \leq \theta \leq \pi \implies \alpha - \beta d \leq \psi \leq \alpha + \beta d$$

Grating Lobes Phenomenon

When more than one peak of the universal pattern projects onto the circle with radius "βd", grating lobe phenomenon occurs.

This means that in addition to the major lobes, there exists other lobes with the same strength.

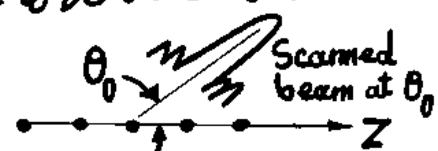


Example: For $\alpha = \pi$ and $2\beta d < 2\pi \Rightarrow d/\lambda < \frac{1}{2}$ no grating lobe occurs.

For $\alpha = 0$ and $\beta d < 2\pi \Rightarrow d/\lambda < 1$ no grating lobe occurs.

How to scan the main beam of Uniform Arrays?

In many practical applications, it is desirable to be able to scan antenna beam in a prescribed direction.



Recall: $\boxed{AF_n = \dfrac{\sin(N\psi/2)}{N\sin(\psi/2)}$: This is maximum when $\psi = 0}$

Recall: $\psi = \beta d\cos\theta + \alpha$: If one desires that AF_n is maximum at $\theta = \theta_0$ angle then

$$\psi = 0 \Rightarrow \boxed{\alpha = -\beta d\cos\theta_0}$$

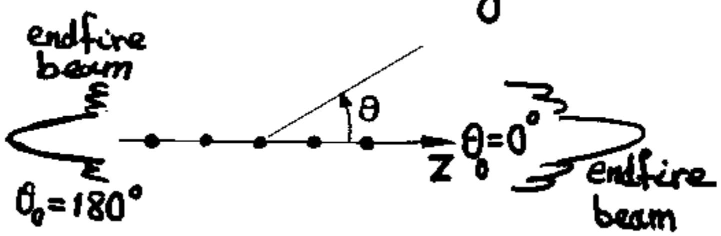
Excitation coefficients to scan the beam at θ_0

$$\boxed{I_n = e^{jn\alpha} = e^{-jn\beta d\cos\theta_0}} \Rightarrow \psi = \beta d(\cos\theta - \cos\theta_0)$$

Note: When $\theta_0 = \pi/2 \Rightarrow \alpha = 0$ (no phase shift) :

Ordinary Endfire Array

The ordinary endfire condition occures when the normalized array factor is maximum at $\theta_0 = 0°$ or $180°$.



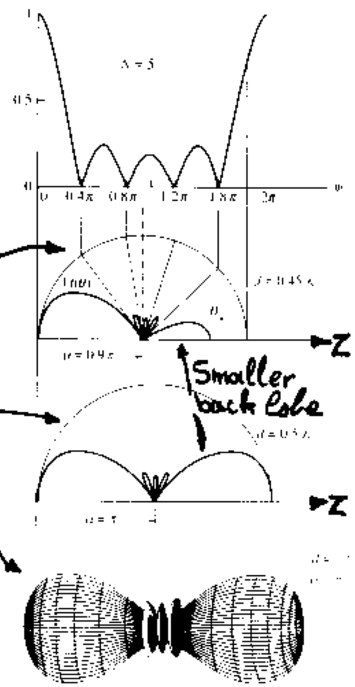
Recall: $\alpha = -\beta d \cos\theta_0$

Then: $\theta_0 = 0° \Rightarrow \alpha = -\beta d$

$\theta_0 = 180° \Rightarrow \alpha = \beta d$

Case II: $\alpha = 0.9\pi$, $d = 0.45\lambda$

Case I: $\alpha = \pi$, $d = 0.5\lambda$



In order to have a dominant endfire lobe eith in $0°$ or $180°$, one needs

$$\boxed{2\beta d \le 2\pi - \pi/N} \quad \text{or} \quad \boxed{d \le \frac{\lambda}{2}\left(1 - \frac{1}{2N}\right)}$$

Hansen-Woodyard Endfire Array

Hansen-Woodyard condition allows one to design endfire arrays with higher directivity than "ordinary" endfire arrays.

Recall: For ordinary endfire $\boxed{\alpha = \pm \beta d}$

Hansen-Woodyard Condition: $\alpha = \pm(\beta d + \delta)$ increase in phase shift

Hansen-Woodyard showed: $\boxed{\delta \approx \frac{2.94}{N-1} \approx \frac{\pi}{N}}$ ⇨ Usually applies to large arrays.

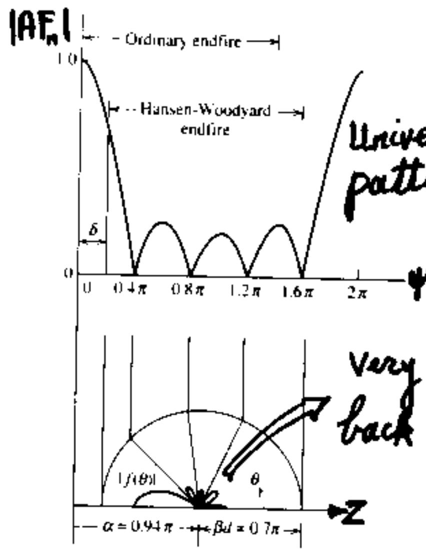
To prevent back lobe: $\alpha = \beta d + \delta < \pi$

Then: $\boxed{\alpha = \pm\left(\beta d + \frac{\pi}{N}\right); \; d < \frac{\lambda}{2}\left(1 - \frac{1}{N}\right)}$

These conditions allow endfire array with higher directivity.

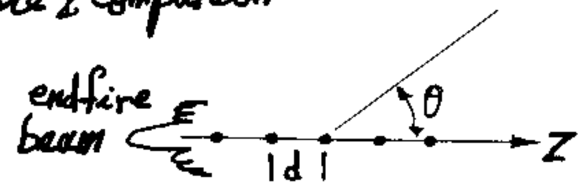
In general, one can use an optimization technique to tailor the performance of an array antenna.

Hansen-Woodyard Endfire Array
Example & Comparison



Universal pattern

Very low back lobes

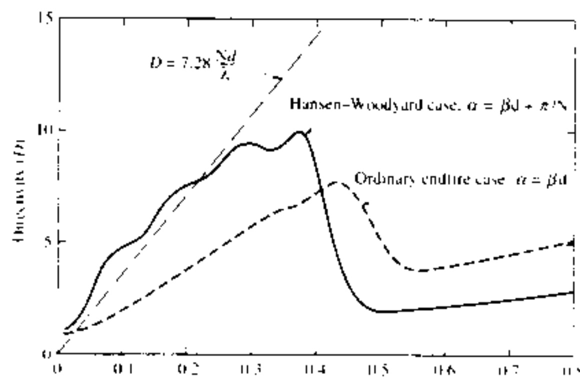


$N=5$

$d=0.37\lambda \Rightarrow \beta d = 0.7\pi$ → Radius of the Circle

$\delta = \frac{\pi}{N} = 0.2\pi$

$\alpha = 0.94\pi$



d/λ

Directivity of Uniform Arrays (1)

Recall: $D = \frac{4\pi}{\Omega_A}$ (directivity) ; Ω_A: Beam solid angle

$\Omega_A = \iint |F(\theta,\phi)|^2 \, d\Omega$; $d\Omega = \sin\theta \, d\theta \, d\phi$; $F(\theta,\phi)$: normalized pattern

Uniform array with isotropic elements: $|F(\theta,\phi)|^2 = |AF_n|^2 = \left|\frac{\sin(N\psi/2)}{N\sin(\psi/2)}\right|^2$; $\psi = \beta d\cos\theta + \alpha$ → independent of ϕ for linear array along z-axis.

Then: $\Omega_A = \int_0^{2\pi} d\phi \int_0^{\pi} |AF_n|^2 \sin\theta \, d\theta = 2\pi \int_0^{\pi} |AF_n|^2 \sin\theta \, d\theta$

or: $\psi = \beta d\cos\theta + \alpha \Rightarrow d\psi = -\beta d \sin\theta \, d\theta \Rightarrow \boxed{\sin\theta \, d\theta = -\frac{1}{\beta d} d\psi}$

Change of variable to ψ: $\boxed{\Omega_A = \frac{2\pi}{\beta d} \int_{-\beta d+\alpha}^{\beta d+\alpha} |AF_n|^2 \, d\psi}$

This is a tough integral; however, it can be done in a closed form!

Directivity of Uniform Arrays (2)

After lengthy manipulations: $\Omega_A = \dfrac{4\pi}{N} + \dfrac{4\pi}{N^2} \displaystyle\sum_{m=1}^{N-1} \dfrac{N-m}{m\beta d}\, 2\cos(m\alpha)\sin(m\beta d)$

Finally: Directivity of Uniform array with isotropic elements.

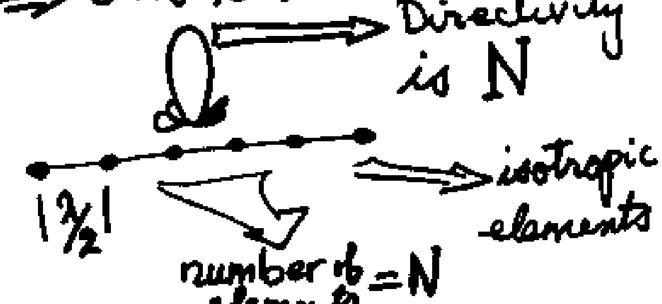
$$\boxed{D = \frac{4\pi}{\Omega_A} = \frac{1}{\dfrac{1}{N} + \dfrac{2}{N^2} \displaystyle\sum_{m=1}^{N-1} \dfrac{N-m}{m\beta d} \cos(m\alpha)\sin(m\beta d)}}$$

Special Cases: Broadside $\lambda/2$ space elements: $\alpha = 0,\ d = \lambda/2$

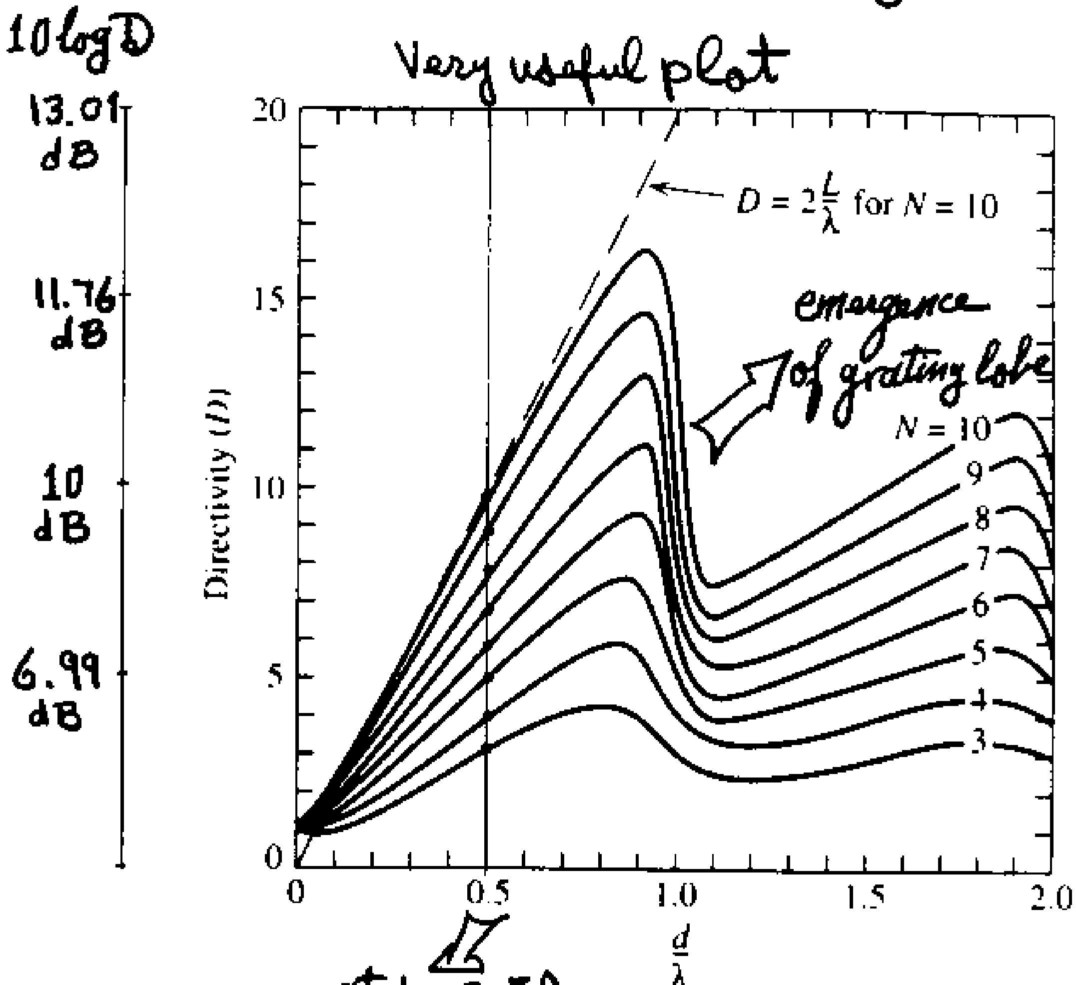
Then $m\alpha = 0 \Rightarrow \cos(m\alpha) = 1$

$m\beta d = m\,\dfrac{2\pi}{\lambda}\,\dfrac{\lambda}{2} = m\pi \Rightarrow \sin(m\beta d) = 0$

Finally: $\boxed{D = \dfrac{1}{\dfrac{1}{N} + 0} = N}$

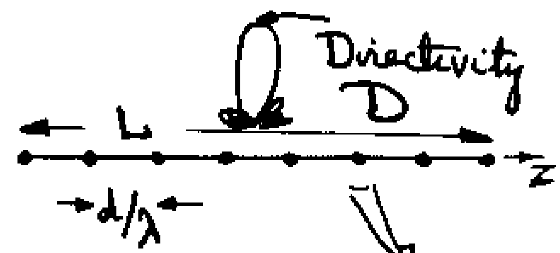


Directivity of broadside arry of isotropic elements vs. d/λ



at $d = 0.5\lambda$

$D = N$ (as expected)



$\alpha = 0$

isotropic elements

N = number of elements

$L/\lambda = (N-1)\dfrac{d}{\lambda}$ array length

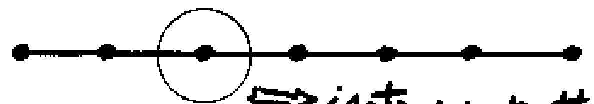
$$D \simeq 2\frac{L}{\lambda} = 2\frac{Nd}{\lambda}$$

when $d = \lambda/2$ $\quad D = N$

This is a good approximation for $d/\lambda < 0.8$

Pattern Multiplication (1)

Array factor can be interpreted as the pattern of an array antenna with isotropic sources.



⇒ isotropic pattern

However, in practice the element pattern is <u>not</u> isotropic and the array antenna is different than its array factor pattern.

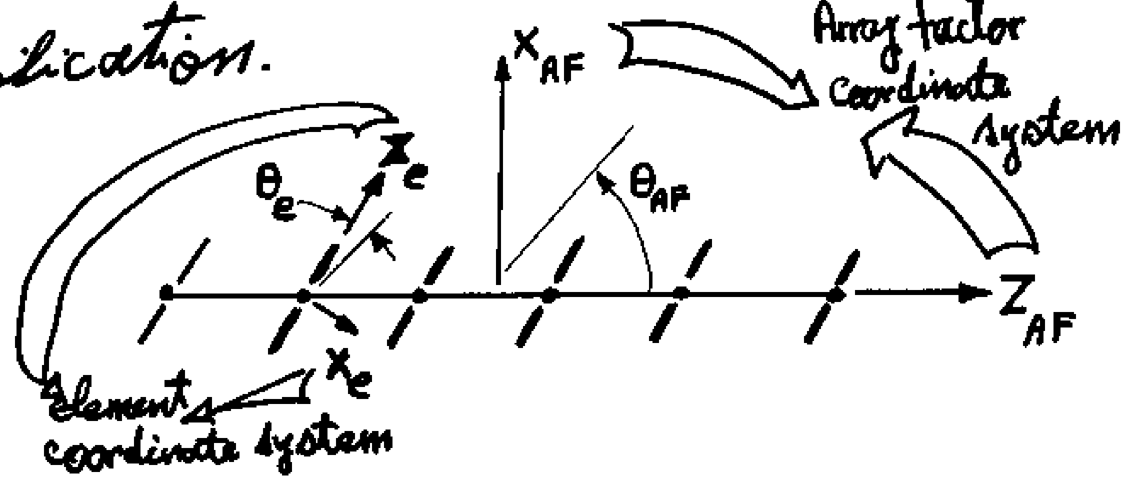
⇒ not isotropic pattern

It has been shown that

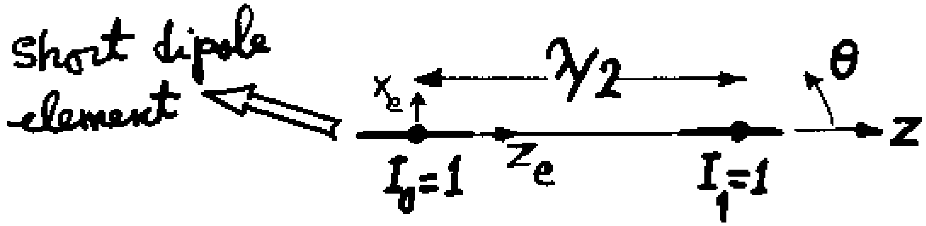
$$\text{Normalized array antenna pattern} = \text{Normalized element pattern} \times \text{Normalized array factor pattern}$$

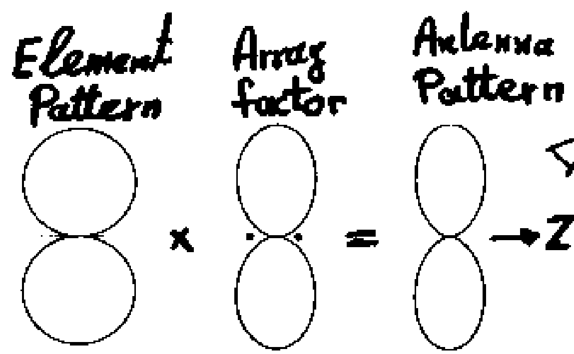
Pattern Multiplication (2)

It must be realized that the most suitable coordinate system to define the array factor may <u>not</u> be the most suitable coordinate system to construct the element pattern. In practice, one may represent the array factor and element pattern in their most suitable coordinates, and then transform one into the other one in order to apply the pattern multiplication.



Pattern Multiplication (3)





This pattern is symmetric about Z.

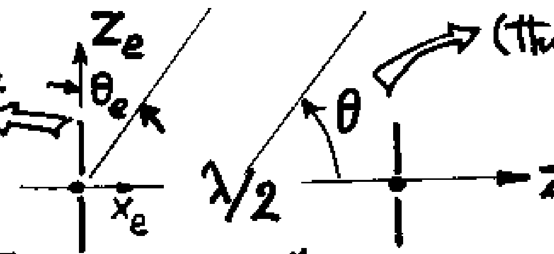
$$\sin\theta \times \cos\left(\frac{\pi}{2}\cos\theta\right) = \sin\theta \cos\left(\frac{\pi}{2}\cos\theta\right)$$

In this case both the array factor coordinates and element pattern coordinate line up.

Pattern Multiplication (4)

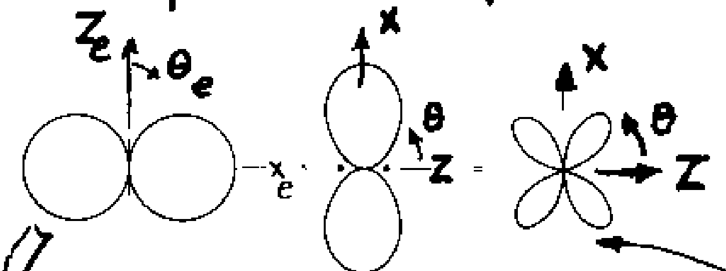
Note: In this case array factor and element coordinates do not line up.

Short dipole element



(this is the same as θ_{AF})

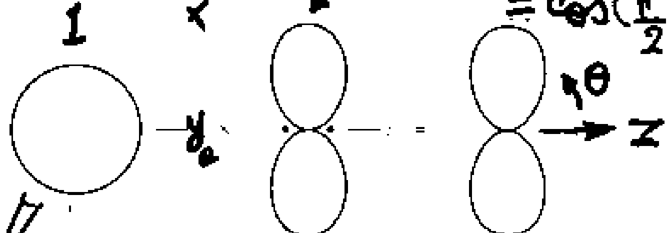
Z (this is the same as Z_{AF})



in x-Z plane ($\phi = 0°$ cut)

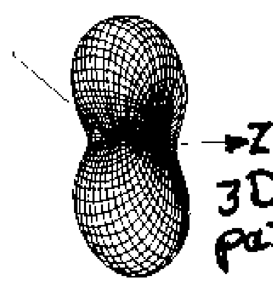
$\sin\theta_e \times \cos(\frac{\pi}{2}\cos\theta) = \sin\theta_e \cos(\frac{\pi}{2}\cos\theta)$ → one needs to relate θ_e to θ.

$1 \times \cos(\frac{\pi}{2}\cos\theta) = \cos(\frac{\pi}{2}\cos\theta)$



in y-Z Plane ($\phi = 90°$ cut)

Note: Element pattern is different in x_e-z_e and y_e-z_e planes

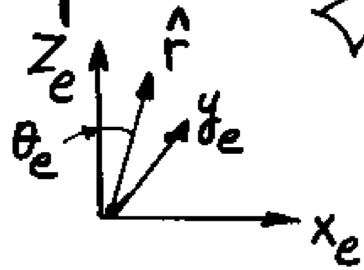


3D pattern

this can be done provided that one knows relative orientation of two coordinate system

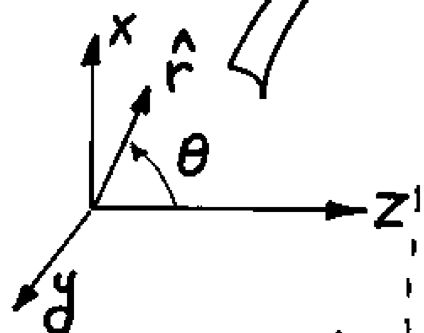
Pattern Multiplication (5)

How to relate (θ_e, ϕ_e) to (θ, ϕ) for a special case:



$$\hat{x}_e = +\hat{z}$$
$$\hat{y}_e = -\hat{y}$$
$$\hat{z}_e = \hat{x}$$

This will be different for different orientations of the coordinates.



$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

Knowing θ & ϕ

$$\boxed{\begin{aligned}\cos\theta_e &= \sin\theta\cos\phi \\ \tan\phi_e &= -\sin\phi\tan\theta\end{aligned}}$$

Recall: $\hat{r} = \sin\theta_e\cos\phi_e\,\hat{x}_e + \sin\theta_e\sin\phi_e\,\hat{y}_e + \cos\theta_e\,\hat{z}_e$

Then: $\hat{r} = \sin\theta_e\cos\phi_e\,\hat{z} + \sin\theta_e\sin\phi_e(-\hat{y}) + \cos\theta_e(\hat{x})$

Compare: $\cos\theta_e = \sin\theta\cos\phi$

$$-\sin\theta_e\sin\phi_e = \sin\theta\sin\phi$$
$$\sin\theta_e\cos\phi_e = \cos\theta$$

Technique can be generalized for any orientations using Eulerian angles.

Directivity of Uniform Arrays with non-isotropic elements

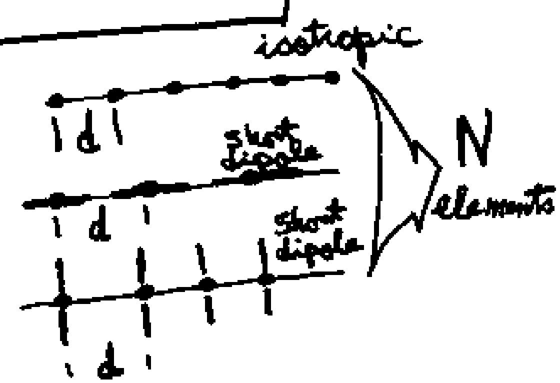
In general, it is almost impossible to obtain the directivity of an antenna in a closed mathematical expression. In these cases, one use numerical integrations to obtain Ω_A and then $D = 4\pi/\Omega_A$.

For some special case, one is able to obtain closed form expressions. Example:

It can be shown:

$$\boxed{D = \frac{1}{\dfrac{a_0}{N} + \dfrac{2}{N^2}\displaystyle\sum_{m=1}^{N-1} \frac{N-m}{m\beta d}\left(a_1 \sin m\beta d + a_2 \cos m\beta d\right)\cos m\alpha}}$$

	a_0	a_1	a_2
Isotropic	1	1	0
— Short dipole	2/3	$2/(m\beta d)^2$	$-2/(m\beta d)$
ǀ short dipole	2/3	$1 - 1/(m\beta d)^2$	$1/(m\beta d)$

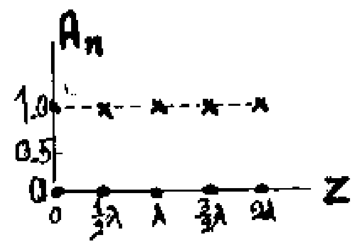


Nonuniformly Excited Equally Spaced Linear Array

- To control the side lobe levels of an array antenna, it becomes necessary to apply nonuniform amplitude excitation coefficients.

$$AF = \sum_{n=0}^{N-1} I_n e^{j\beta d\cos\theta} = \sum_{n=0}^{N-1} A_n e^{jn\psi} \quad ; \quad I_n = A_n e^{jn\alpha}$$

A_n → nonuniform amplitude; A_n → real positive number

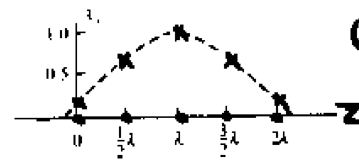


(a) Uniform

Uniform

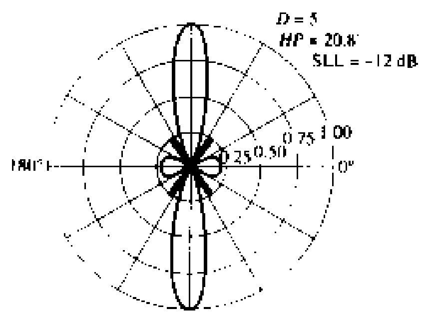
(b) Triangular

Triangular

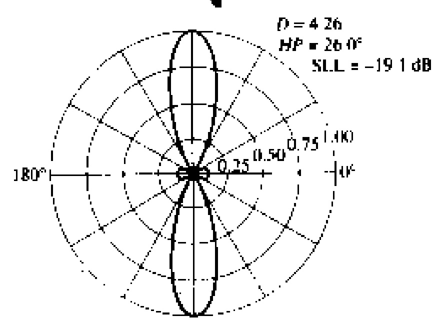


(c) Binomial

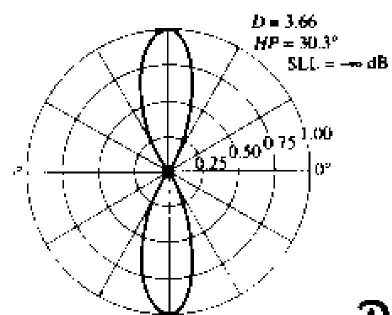
Binomial



(a) Uniform



(b) Triangular



(c) Binomial

observation:
As amplitude tapers towards the ends of the array antenna.

- Directivity → down
- Sidelobe → down
- Beamwidth → widens

Binomial Coefficient Array (1)

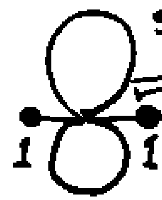
- Linear arrays with binomial coefficients create <u>no side lobes</u> provided that $d \le \lambda/2$.

$$AF = \sum_{n=0}^{N-1} A_n e^{jn\psi} = \sum_{n=0}^{N-1} \underbrace{A_n}_{\text{real number}} Z^n \qquad \text{Define: } Z = e^{j\psi}$$

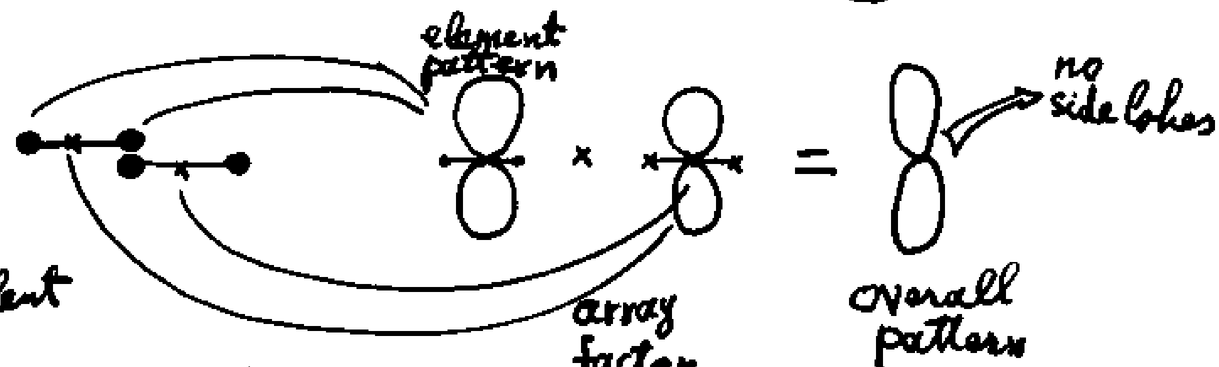
this is an $(N-1)$ order polynomial in Z.

For two elements: $AF = 1 + Z$

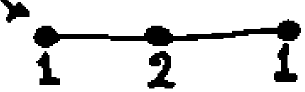
Note: For broadside ($\alpha = 0$) and $d < \lambda/2$ $\Longrightarrow$



observe:



$(Z+1)(Z+1) = 1 + 2Z + Z^2$



Binomial Coefficient Array (2)

3-element binomial : $AF = (1+Z)(1+Z) = (1+Z)^2$

1 2 1

5-element binomial : $AF = (1+Z)^{\overset{5-1}{4}} = 1 + 4Z + 6Z^2 + 4Z^3 + Z^4$

1 4 6 4 1

N-element binomial : $AF = (1+Z)^{N-1}$

Pascal's triangle to create binomial coefficients :

N	
1	1
2	1 1
3	1 2 1
4	1 3 3 1
5	1 4 6 4 1
⋮	....

Directivity of Linear Arrays (1)

Previously, the directivity of uniform arrays were derived. Here, the directivity is obtained for a general case of linear arrays.

Recall: $\boxed{D = \frac{4\pi}{\Omega_A} \quad ; \quad \Omega_A = \iint |F_n(\theta,\phi)|^2 \, d\Omega}$

Let $\alpha_n = -\beta z_n \cos\theta_0$

For a linear array along the z axis

$$F(\theta) = \sum_{n=0}^{N-1} A_n e^{j\alpha_n} e^{j\beta z_n \cos\theta} \Rightarrow F_n(\theta) = AF_n = \frac{\sum_{n=0}^{N-1} A_n e^{j\alpha_n} e^{j\beta z_n \cos\theta}}{\sum_{n=0}^{N-1} A_n}$$

No ϕ dependence

Normalization factor, for beam peak in θ_0 direction.

Then: $\boxed{\Omega_A = \int_0^{2\pi}\int_0^{\pi} |F_n(\theta)|^2 \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} |F_n(\theta)|^2 \sin\theta \, d\theta}$

Directivity of Linear Arrays (2)

$$|F_n(\theta)|^2 = F_n(\theta)\, F_n^{*}(\theta) = \frac{\sum_{n=0}^{N-1} A_n e^{j\alpha_n} e^{j\beta z_n\cos\theta}}{\sum_{n=0}^{N-1} A_n} \cdot \frac{\sum_{p=0}^{N-1} A_p e^{-j\alpha_p} e^{-j\beta z_p\cos\theta}}{\sum_{n=0}^{N-1} A_n}$$

conjugate

because of conjugate

or by proper grouping:

$$|F_n(\theta)|^2 = \frac{1}{\left(\sum_{n=0}^{N-1} A_n\right)^2} \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} A_n A_p e^{j(\alpha_n-\alpha_p)} e^{j\beta(z_n-z_p)\cos\theta}$$

therefore:

an easy integral to do in closed form.

$$\Omega_A = \frac{2\pi}{\left(\sum_{n=0}^{N-1} A_n\right)^2} \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} A_n A_p e^{j(\alpha_n-\alpha_p)} \int_0^{\pi} e^{j\beta(z_n-z_p)\cos\theta} \underbrace{\sin\theta\, d\theta}_{-d(\cos\theta)}$$

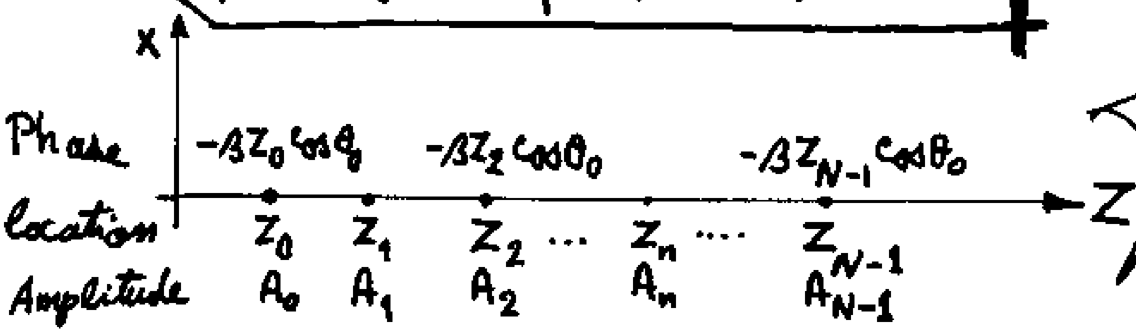
Directivity of Linear Arrays (3)

Observe: $$\int_0^{\pi} -e^{j\beta(z_n-z_p)\cos\theta}\, d(\cos\theta) = \frac{-1}{j\beta(z_n-z_p)}\Big[\underbrace{e^{-j\beta(z_n-z_p)} - e^{j\beta(z_n-z_p)}}_{-2j\sin[\beta(z_n-z_p)]}\Big]$$

Finally: $$\boxed{D = \frac{4\pi}{\Omega_A} = \frac{\left(\sum_{n=0}^{N-1} A_n\right)^2}{\sum_{n=0}^{N-1}\sum_{p=0}^{N-1}\left[A_n A_p \frac{\sin[\beta(z_n-z_p)]}{\beta(z_n-z_p)} \cdot e^{j(\alpha_n-\alpha_p)}\right]}}$$

Note: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Very useful result for elements at any position z_n, current amplitude A_n and phase of $\alpha_n = -\beta z_n \cos\theta_0$



N element isotropic along Z axis

Directivity of Linear Arrays (4)
Special Cases

Broadside and equally spaced: $\alpha_n = 0, \; z_n = nd$

then:

$$\boxed{D = \frac{\left(\sum_{n=0}^{N-1} A_n\right)^2}{\sum_{n=0}^{N-1}\sum_{p=0}^{N-1} A_m A_p \frac{\sin[(m-p)\beta d]}{(m-p)\beta d}}}$$

Let: $\boxed{d = \lambda/2, \lambda, \ldots}$ multiple of a half-wavelength $\Rightarrow \beta d = \pi, 2\pi, \ldots$

Recall: $\sin[(m-p)\left|\begin{smallmatrix}\pi\\2\pi\\\vdots\end{smallmatrix}\right|] = 0$ for $m \neq p$ for $m = p$ $\frac{\sin[(m-p)\beta d]}{(m-p)\beta d} \to 1$

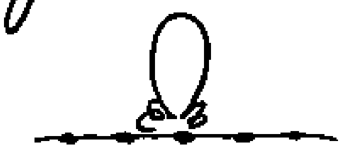
therefore $\boxed{D = \frac{\left(\sum_{n=0}^{N-1} A_n\right)^2}{\sum_{n=0}^{N-1} (A_n)^2}}$ Very interesting result $\Longrightarrow D = N$ when $A_0 = A_1 = \cdots = A_{N-1}$ $\Longrightarrow$ similar result as before.

Scanned Beams and Avoidance of Grating Lobes (1)

When there is a uniform progressive phase $\alpha_n = -\beta z_n \cos\theta_0$ in the phase of elements, the beam is steered in $\theta = \theta_0$ direction.

To aviode grating lobes one needs: $\boxed{d < \dfrac{\lambda}{1 + |\cos\theta_0|}}$ Very important result in practice

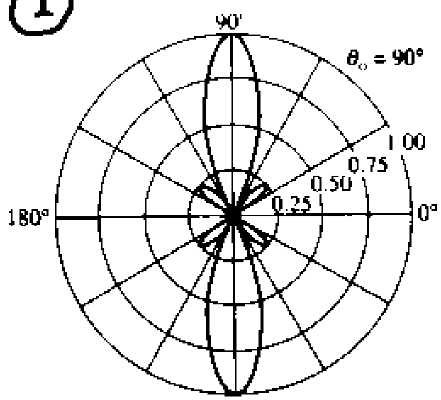
This means that as one scans farther away from the broadside direction, one needs elements more closely positioned.

For example at broadside $\theta_0 = 90^\circ \Rightarrow d < \lambda$ 

at <u>end fire</u> $\theta_0 = 0^\circ$ or $180^\circ \Rightarrow d < \lambda/2$ 

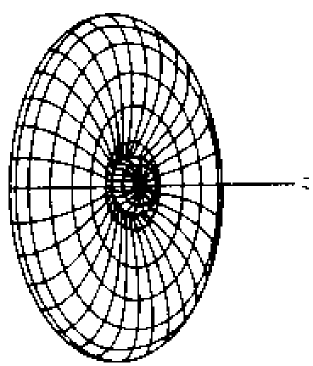
Scanned Beams (2): Example

1



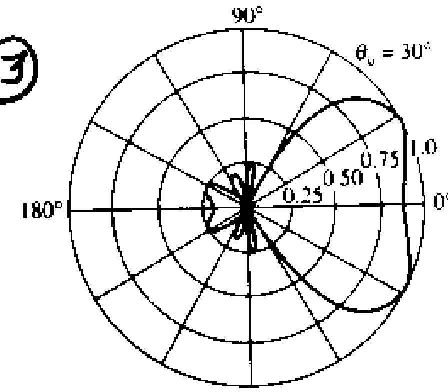
(a) $\theta_o = 90°$

Broadside



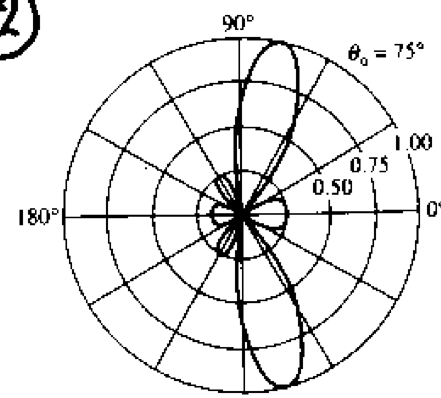
(b) $\theta_o = 90°$

3



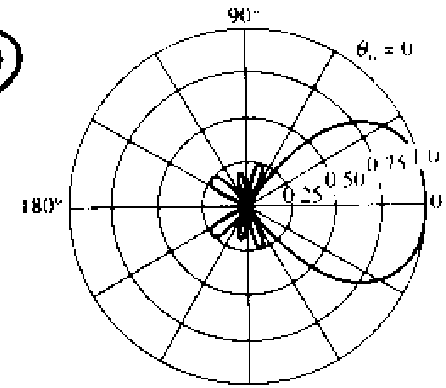
(d) $\theta_o = 30°$ (bifurcated pattern)

2



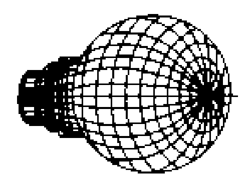
(c) $\theta_o = 75°$

4



(e) Endfire ($\theta_o = 0°$)

Endfire



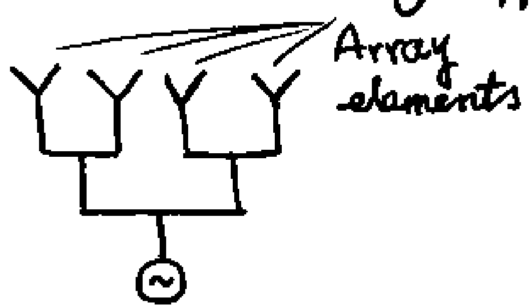
(f) Endfire (θ_o = [illegible])

Note: Beam broadens as it scans.

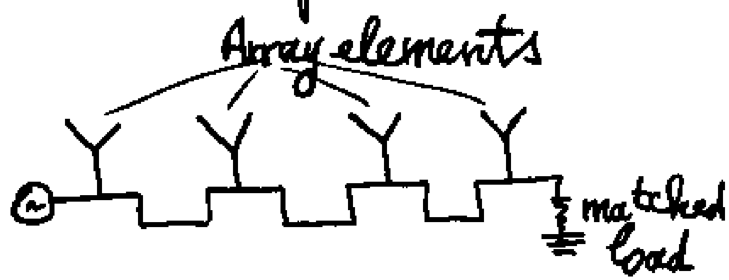
Feed Networks

The hardware connecting elements of an array to the transmitter (or receiver) are called "feed network" or "beam-forming Network (BFN)".

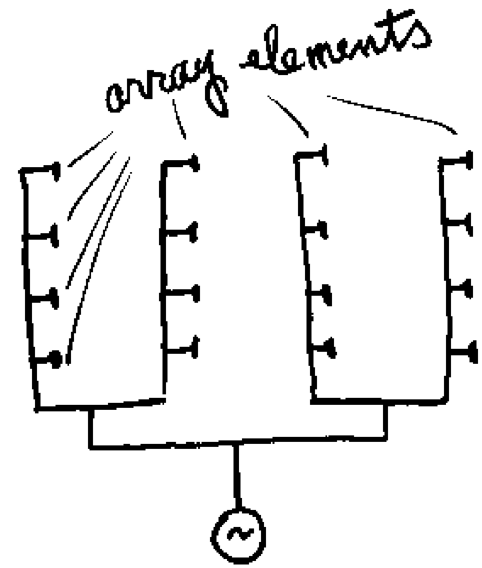
BFN can take many different forms. Examples:



Parallel or Corporate BFN



Series BFN



Parallel-series BFN

Each BFN configuration has its own advantages & disadvantages.

Array Factor : this is a scalar quantity (1)

Planar Array

Array factor plays paramont role in controling the radiation characteristics of array antennas.

Array Factor : $\boxed{AF = \sum_{i=1}^{N} I_i \, e^{j\beta \hat{r}\cdot\vec{r}_i}}$

$\hat{r}$ → observation direction

$\vec{r}_i$ → ith element position

I_i → ith element complex excitation coefficient

β → propagation constant $\beta = \frac{2\pi}{\lambda}$, λ → Wavelength

$I_i = |I_i| e^{j\varphi_i}$

$|I_i|$ → magnitue

φ_i → phase

Wavelength ← $\lambda = \frac{c}{f}$, c → speed of light, f → frequency

$$\begin{cases} \hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \\ \vec{r}_i = x_i\,\hat{x} + y_i\,\hat{y} + z_i\,\hat{z} \end{cases}$$

$$\boxed{\hat{r}\cdot\hat{r}_i = x_i\sin\theta\cos\phi + y_i\sin\theta\sin\phi + z_i\cos\theta}$$

YRS-44

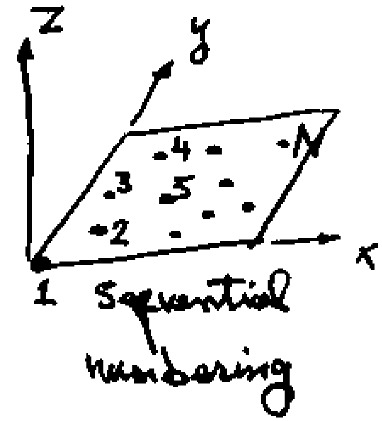
Array factor for Planar Array (2)

For a planar array residing in x-y plane

$$\vec{r}_i = x_i\,\hat{x} + y_i\,\hat{y} \quad \text{no } \hat{z} \text{ component}$$

then :

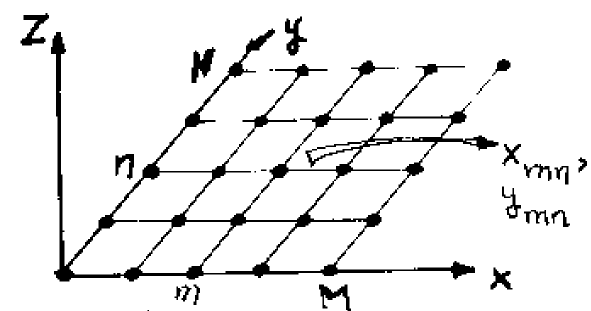
$$\hat{r}\cdot\vec{r}_i = x_i \sin\theta\cos\phi + y_i \sin\theta\sin\phi$$



When array elements line up along x and y directions one may use double indexing

$$\vec{r}_{mn} = x_{mn}\,\hat{x} + y_{mn}\,\hat{y}$$

$$\hat{r}\cdot\vec{r}_{mn} = x_{mn}\sin\theta\sin\phi + y_{mn}\sin\theta\cos\phi$$



Array factor in double summation :

$$AF(\theta,\phi) = \sum_{n=1}^{N}\sum_{m=1}^{M} \underbrace{A_{mn}\, e^{j\alpha_{mn}}}_{I_{mn}}\; e^{j\beta\,\hat{r}\cdot\vec{r}_{mn}}$$

Array Factor for Planar Array : Scanned Beam (3)

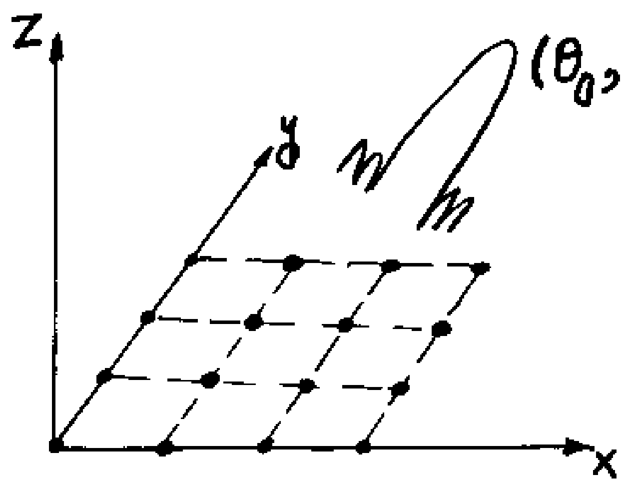
For a planar array with its beam scanned in $\hat{r}_0$ direction

$$\hat{r}_0 = \sin\theta_0 \cos\phi_0 \hat{x} + \sin\theta_0 \sin\phi_0 \hat{y} + \cos\theta_0 \hat{z}$$

$$\alpha_{mn} = -\beta \hat{r}_0 \cdot \vec{r}_{mn} = -\beta x_{mn} \sin\theta_0 \cos\phi_0 - \beta y_{mn} \sin\theta_0 \sin\phi_0$$

Then :

$$\boxed{AF(\theta,\phi) = \underbrace{\sum_{n=1}^{N} \sum_{m=1}^{M}}_{\text{Total of } M \cdot N \text{ elements}} A_{mn}\, e^{j\beta \overbrace{(\hat{r}-\hat{r}_0)\cdot \vec{r}_{mn}}^{\text{the beam is steered at } \hat{r}=\hat{r}_0 \text{ direction.}}}}$$



Total of M·N elements

the beam is steered at $\hat{r}=\hat{r}_0$ direction.

Array Factor for Planar Array: Separable excitations (4)

When all elements parallel to x-axis have the same excitation coefficients and all elements parallel to y-axis have the same excitation coefficients then the array excitation coefficients is <u>separable</u>, namely, $I_{mn} = I_{xm} I_{yn}$

then: $$AF = \sum_{m=1}^{M} I_{xm} e^{j x_m \sin\theta\cos\phi} \cdot \sum_{n=1}^{N} I_{yn} e^{jK_n \sin\theta \sin\phi}$$

Product of two linear array factor

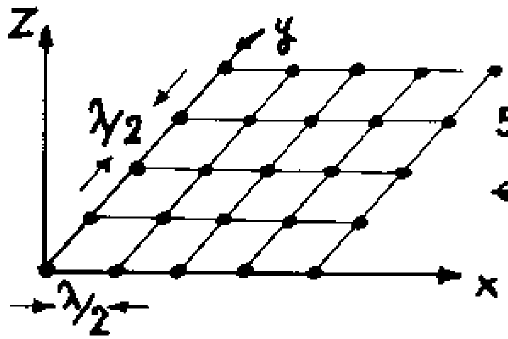
element spacing

For special case of $x_m = md_x$; $y_n = nd_y$ with progressive phase:

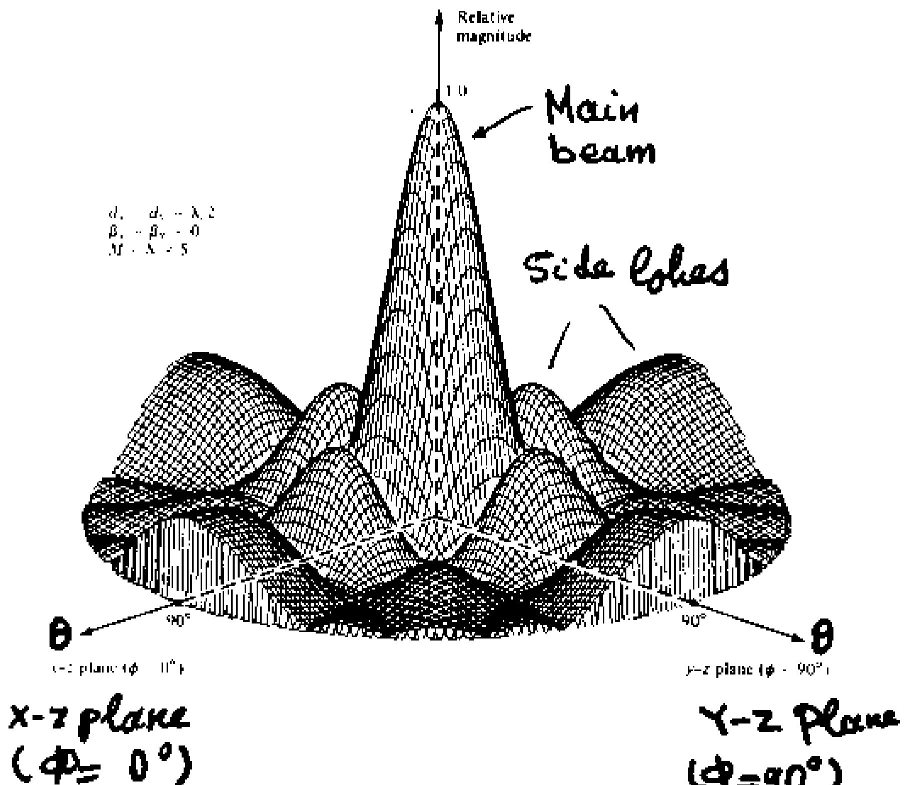
$$\boxed{AF_n(\theta,\phi) = \left\{ \frac{\sin\left(\frac{M}{2}\psi_x\right)}{M \sin\frac{\psi_x}{2}} \right\} \left\{ \frac{\sin\left(\frac{N}{2}\psi_y\right)}{N \sin\frac{\psi_y}{2}} \right\}} \; ; \quad \boxed{\begin{aligned} \psi_x &= \beta d_x \sin\theta\cos\phi + \alpha_x \\ \psi_y &= \beta d_y \sin\theta\sin\phi + \alpha_y \end{aligned}}$$

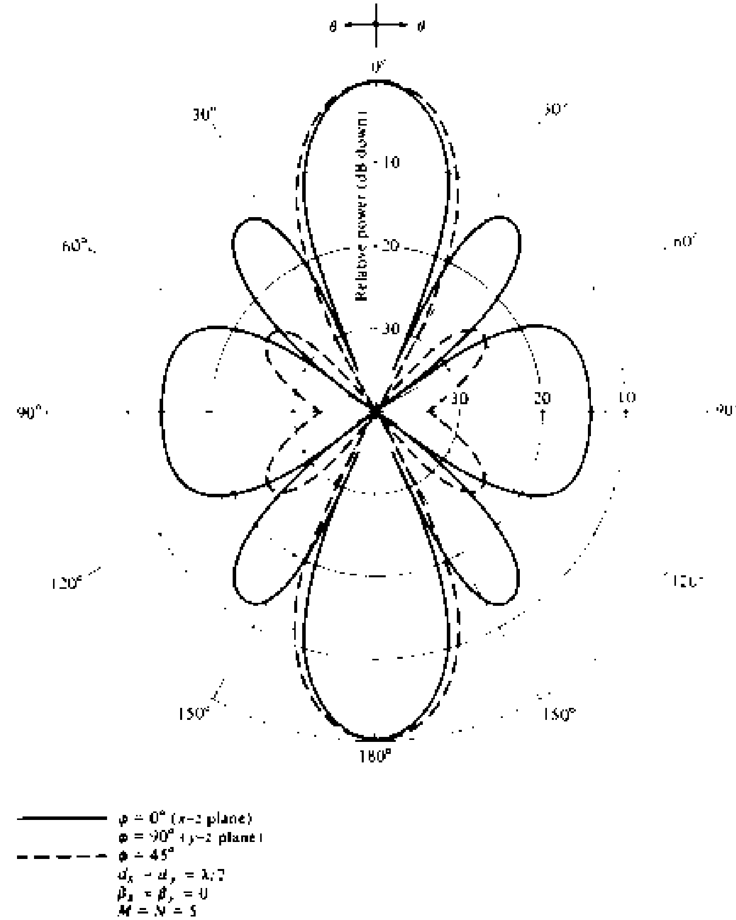
Very useful result for separable arrays

Planar Array: Example (5)



5×5 isotropic element planar array





Frequency Independent Antennas (1)

Classification:

Broadband Antennas Bandwidth $\quad \dfrac{f_U}{f_L} \geq 2:1$ $\quad \begin{cases} \text{Helix} \\ \text{Traveling wave} \\ \text{Biconical (short)} \\ \vdots \end{cases}$

Frequency Independent Antennas Bandwidth $\quad \dfrac{f_U}{f_L} \geq 10:1$ $\quad \begin{cases} \text{Spiral} \\ \text{Log Periodic} \\ \text{Biconical (infinite)} \\ \vdots \end{cases}$

The ultimate configuration of a frequency independent independent antenna has constant pattern, impedance, polarization and phase center with frequency.

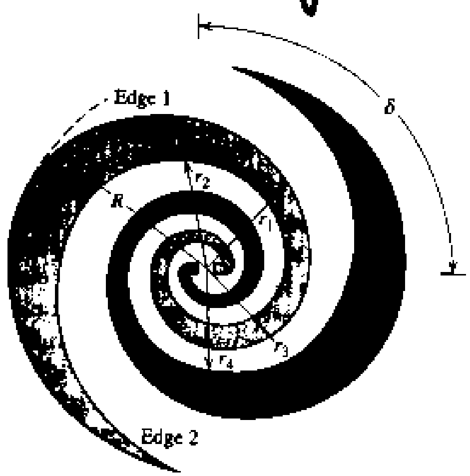
Frequency Independent Antennas (2)

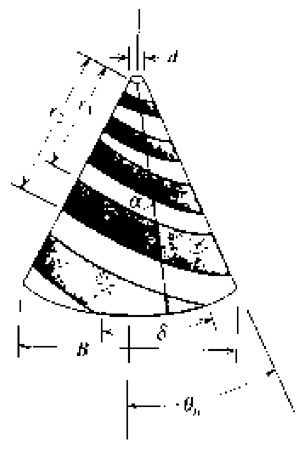
The most successful design approach for frequency independent antennas evolved based on Rumsey's principle.

" Rumsey's principle is that the impedance and pattern properties of an antenna will be frequency independent if the antenna shape is specified only in terms of angles."

Example:

equiangular spiral antenna

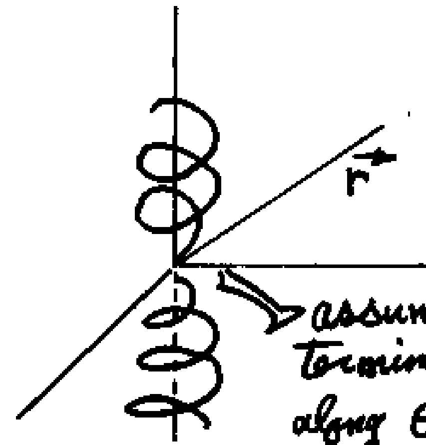




Conical equiangular spiral antenna

Rumsey's Concept (1)

Surface of edge description : $r = F(\theta, \phi)$ → function of angles



- To scale the antenna to a frequency K times lower, the antenna physical dimensions should be K times greater.

or. $$r' = K F(\theta, \phi)$$

- It is required that the new and old surfaces be identical.

this can be achieved by rotation in ϕ.

Therefore $$K F(\theta, \phi) = F(\theta, \phi + C)$$ → angle of rotation

Rumsey's Concept. (2)

Differentiate with respect to C :

$$\frac{d}{dC}[KF(\theta,\phi)] = \frac{dK}{dC}F(\theta,\phi) = \frac{\partial}{\partial C}[F(\theta,\phi + C)]$$
$$= \frac{\partial}{\partial(\phi + C)}[F(\theta,\phi + C)]$$

Differentiate with respect to ϕ :

$$\frac{\partial}{\partial\phi}[KF(\theta,\phi)] = K\frac{\partial F(\theta,\phi)}{\partial\phi} = \frac{\partial}{\partial\phi}[F(\theta,\phi + C)]$$
$$= \frac{\partial}{\partial(\phi + C)}[F(\theta,\phi + C)]$$

Obtain :

$$\frac{dK}{dC}F(\theta,\phi) = K\frac{\partial F(\theta,\phi)}{\partial\phi}$$

$$\frac{1}{K}\frac{dK}{dC} = \frac{1}{r}\frac{\partial r}{\partial\phi}$$

independant of θ and ϕ

find r?

Finally :

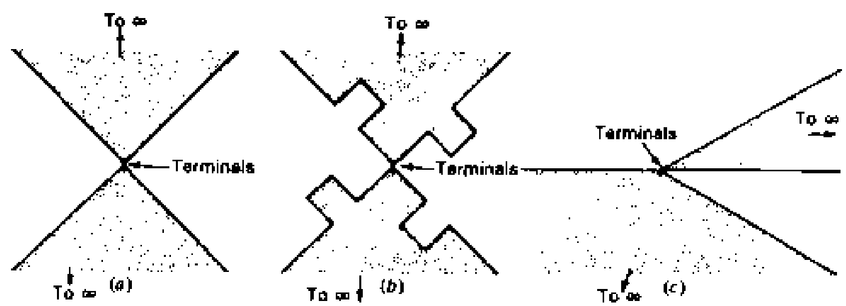
$$\boxed{\begin{array}{l} r = F(\theta,\phi) = e^{a\phi}f(\theta) \\ \text{where} \quad a = \frac{1}{K}\frac{dK}{dC} \end{array}}$$

arbitrary

Frequency Independent Antennas (3)

- In addition to angle emphasis design, the feature of geometrical self-complementarity also leads to frequency independent antenna.
- A self-complementry planar antenna has a metal area congruent to the open area. This means that the two areas can be brought into coincidence by rigid motion.

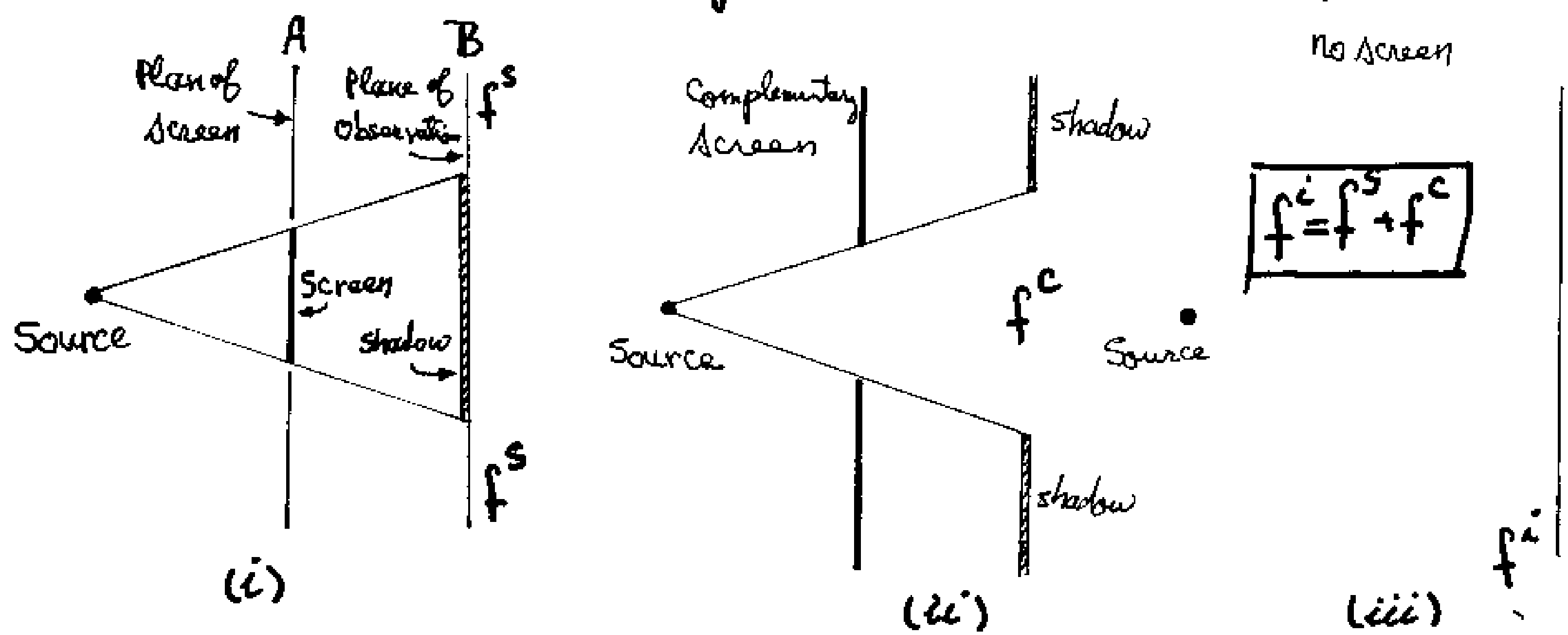
Example:



Self-Complementary Antennas

Babinet's Principle in Optics (0)

Babinet's principle in optics: "The field at any point behind a plane having a screen, if added to the field at the same point when the complementary screen is substituted, is equal to the field at the point when no screen is present."



Babinet's Principle in EM (1)

Babinet's principle in optics primarily deals with absorbing screens and does <u>not</u> consider polarization.

An extension of Babinet's principle in electromagnetics (EM) was suggested by Booker.

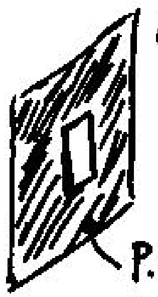
Note: The detailed mathematical derivation is given in a book by Elliott. Here, we only discuss the main features.

Babinet's Principle in EM (2)

source — medium — observation

(i) $\vec{J}\uparrow$ $\quad \epsilon, \mu \quad$ $\overset{P}{\bullet}$ $\vec{E}^i, \vec{H}^i; \eta = \sqrt{\mu/\epsilon}$

(ii) $\vec{J}\uparrow$

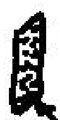


ϵ, μ

P.e.c. screen (infinite)

$\overset{P}{\bullet}$ $\vec{E}^e, \vec{H}^e; \eta = \sqrt{\mu/\epsilon}$

(iii) $\vec{J}\uparrow$

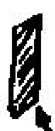


ϵ, μ

P.m.c. screen (finite)

$\overset{P}{\bullet}$ $\vec{E}^m, \vec{H}^m; \eta = \sqrt{\mu/\epsilon}$

(iv) $\vec{M}\uparrow$



μ, ϵ

P.e.c. screen (finite)

$\overset{P}{\bullet}$ $\vec{E}^d, \vec{H}^d; \eta_d = \sqrt{\epsilon/\mu}$

$$\vec{E}^i = \vec{E}^e + \vec{E}^m$$
$$\vec{H}^i = \vec{H}^e + \vec{H}^m$$

$$\boxed{\begin{aligned}\vec{E}^i &= \vec{E}^e + \vec{H}^d \\ \vec{H}^i &= \vec{H}^e - \vec{E}^d\end{aligned}}$$

Dual situation

$\vec{J} \to \vec{M}$

P.m.c. screen $\to$ P.e.c. screen

$\epsilon \to \mu$

$\mu \to \epsilon$

$\vec{E}^m \to \vec{H}^d$

$\vec{H}^m \to -\vec{E}^d$

Screens in cases (ii) and (iv) create complimentary structures

Babinet's Principle in EM (3)

It can be shown that

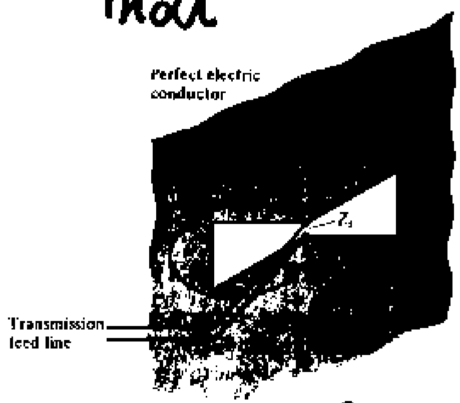
$$Z_s \, Z_c = \frac{\eta^2}{4}$$

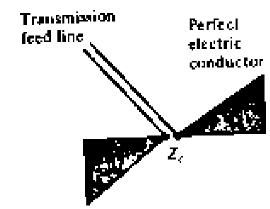
free space impedance $\sqrt{\frac{\mu}{\epsilon}} = 120\pi \approx 377\,\Omega$

Frequency independent

Impedance of Screen like antenna

Impedance of its complementary structure





For self-complementary structures: $Z_s = Z_c$

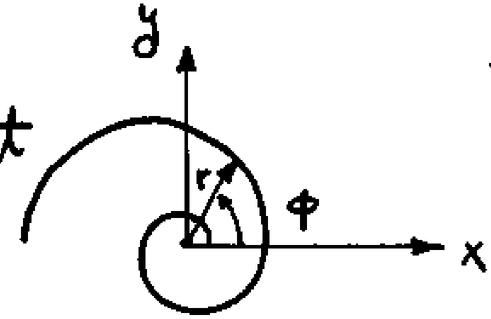
then:

$$\boxed{Z_s = Z_c = \frac{\eta}{2} = 188.5\,\Omega}$$

Frequency independent

Equiangular Spiral Antenna (1)

Equation: $r = r_0 e^{a\phi}$ only angle dependent



this is a right-handed spiral for +z direction.

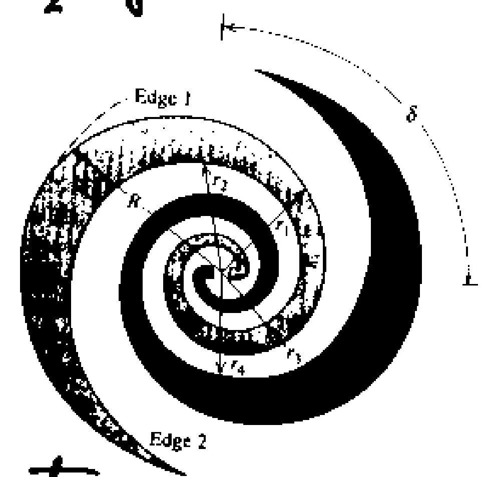
a is a constant controlling the flare rate of the spiral.

this "angle" dependent curve can be used to create an antenna with large f_U/f_L.

$$r_1 = r_0 e^{a\phi} \; ; \; r_2 = r_0 e^{a(\phi-\delta)}$$

$$r_3 = r_0 e^{a(\phi-\pi)}$$

$$r_4 = r_0 e^{a(\phi-\pi-\delta)}$$



For $\delta = 90°$, one obtains a self-complementary structure.

Equiangular Spiral Antenna (2)

The impedance, pattern and polarization of this ant remain nearly constant over a wide range of frequency.

What antenna parameters control the bandwidth?:

(i) the feed point at the center

(ii) the overall radius

(iii) the flare rate

Expansion ratio: $\epsilon = \dfrac{r(\phi+2\pi)}{r(\phi)} = \dfrac{r_0 e^{a(\phi+2\pi)}}{r_0 e^{a\phi}} = e^{2\pi a}$ increase of the radius for one turn of the spiral.

Typical value of $\boxed{\epsilon = 4 \Rightarrow a = 0.221}$

Equiangular Spiral Antenna (3)

- Upper frequency : $f_U = \dfrac{c}{\lambda_U}$; $\lambda_U = 2\pi r_0$ → minum radius

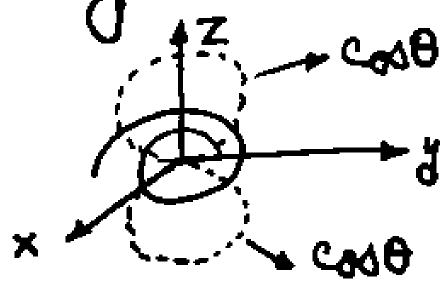
- Lower frequency : $f_L = \dfrac{c}{\lambda_L}$ (c → speed of light) ; $\lambda_L = 2\pi R$ → overall radius

Bandwidth of 8:1 are typical.
Up to 40:1 can be obtained.

- Turns : Typically one-half to three turns are used. One and one-half turn ($\phi = 3\pi$) is about optimum.

- Impedance : Due to self complementary $Z \approx 188.5 + j0$

- Radiation pattern & polarization : Bi-directional radiation

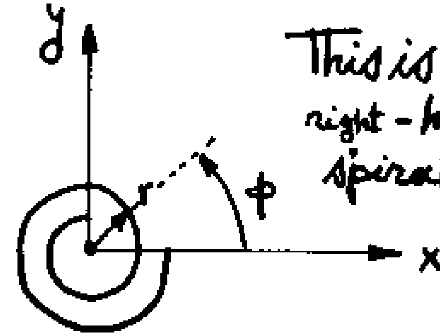


(Right hand) CP

(Left hand) CP

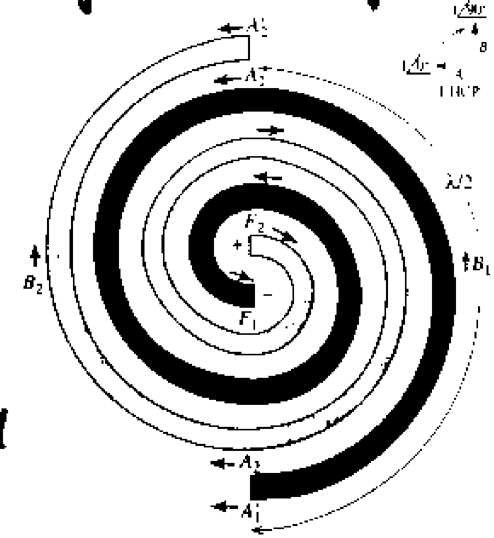
Archimedean Spiral Antenna (1)

Equation: $r = r_0 \phi$ Only angle dependent



This is a right-handed spiral for +Z direction.

Note: For this curve, r is linearly proportional to the angle rather than exponential dependence. This results in much slower flares.

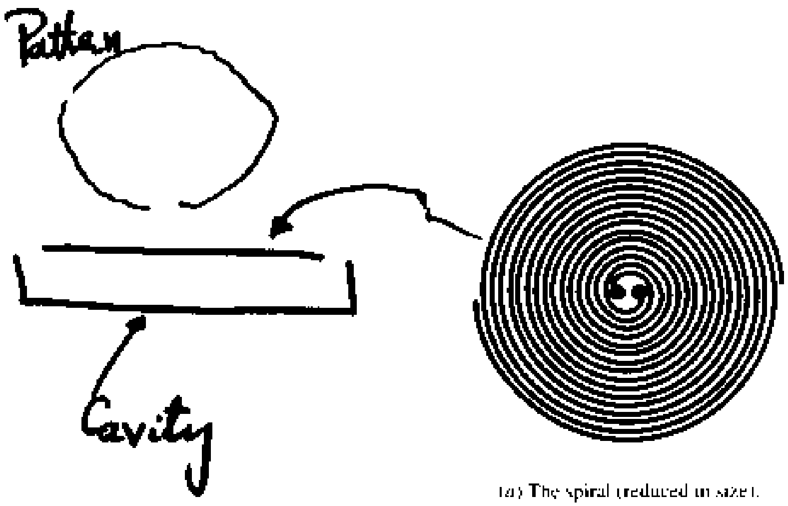


This "angle" dependent curve can be used to create an antenna with large f_U/f_L.

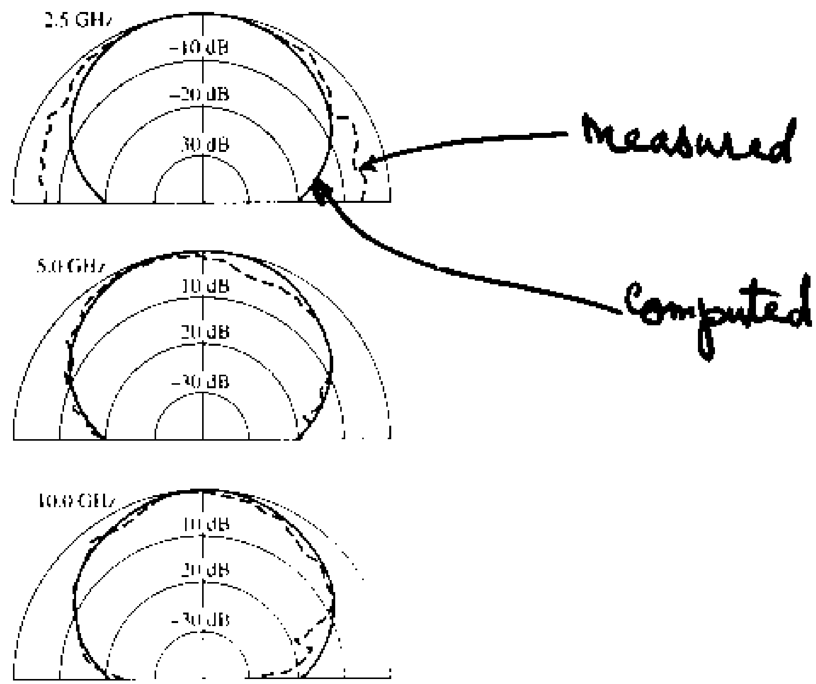
This is left-hand for +Z direction.

Radiation characteristics of this antenna has certain similarity to the equiangular spiral antenna

Cavity-Backed Archimedean Spiral Antenna



(a) The spiral (reduced in size).

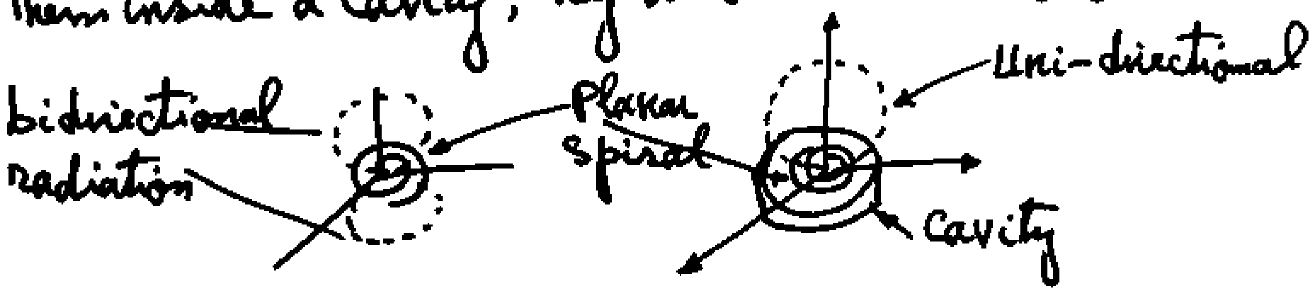


4:1
bandwidth

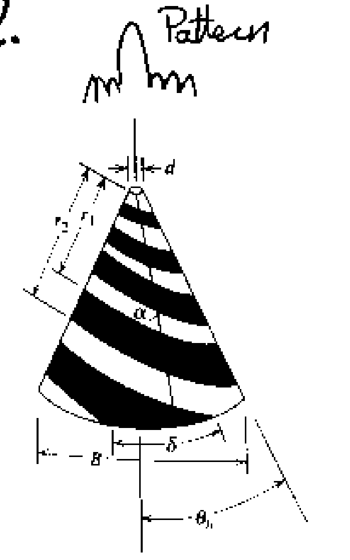
Approximately 5 dB gain.

Conical Equiangular Spiral Antenna

Planar spiral antennas have bi-directional radiation. By positioning them inside a cavity, they become uni-direction.



Another way to get unidirection radiation is to use nonplanar forms of spiral antenna, such as, conical spiral.



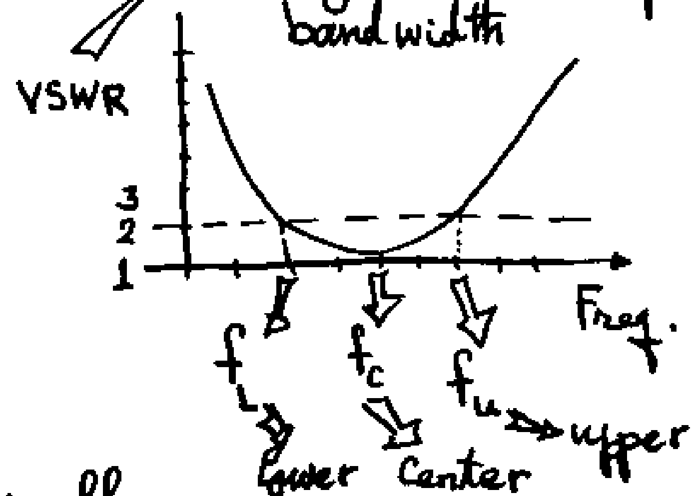
Equation: $r = e^{(a \sin\theta_h)\phi}$ → only angle dependent

This "angle" dependent curve can be used to create an antenna.

Broadband Antennas (1)

There are many applications that it is extremely important to use antenna with broad bandwidth.

Different characteristics may be used to define bandwidth



Definition :

$$f_c = \frac{f_u + f_L}{2}$$

- Bandwidth as a percent of the center freq.

$$B_p = \frac{f_U - f_L}{f_c} \times 100\%$$

usually used for small bandwidth antennas.

- Bandwidth as a ratio

$$B_r = \frac{f_U}{f_L}$$

usually used for large bandwidth antennas.

Broadband Antennas (2)

- Classification of Broadband Antennas: In practice, antennas whose impedance and pattern do not change significantly over about an octave ($f_U/f_L = 2$) or more are considered to be broadband.

- Generally speaking, resonant antennas are <u>Not</u> broadband. Their bandwidth can vary from 0.5% to 30%.

- Broadband Antennas:
  - Traveling wave antennas
  - Helical antennas
  - Biconical antennas
  - Spiral antennas
  - Log periodic antennas
  - etc.

Broadband Antennas: Traveling Wave (1) (only 10% to 20%)

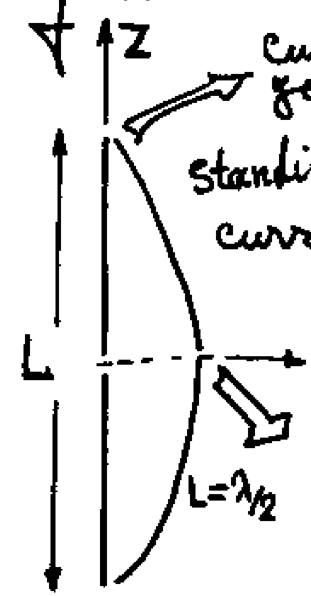
Question: What makes the dipole antenna Not to be broadband?

Answer: The stablishment of the "standing wave" current.

What causes standing wave current? : Reflection from the end points of the antenna.

Recall: $I(z) = I_m \sin\left[\beta\left(\frac{L}{2} - Z\right)\right]$

Rewrite: $I(z) = \frac{I_m}{2j} e^{j\beta L/2} \left\{ \underbrace{e^{-j\beta Z}}_{\text{outward wave}} - \underbrace{e^{-j\beta L} e^{+j\beta Z}}_{\text{reflected wave}} \right\}$

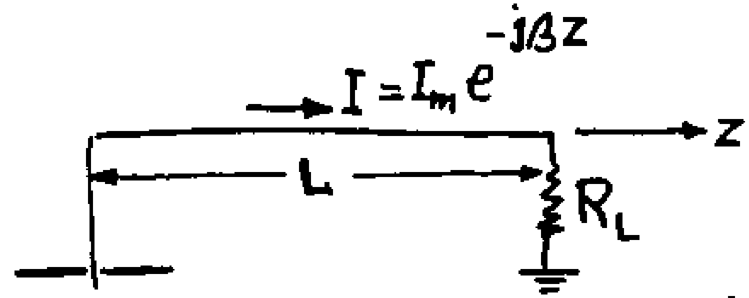


$Z_A = \frac{V}{I_m \sin\frac{\beta L}{2}}$ (Impedance) → Strong function of frequency

Observation: Supression of the reflected wave will enhance bandwidth. How?

Broadband Antennas: Traveling Wave (2)

Long wire antennas terminated to a load:



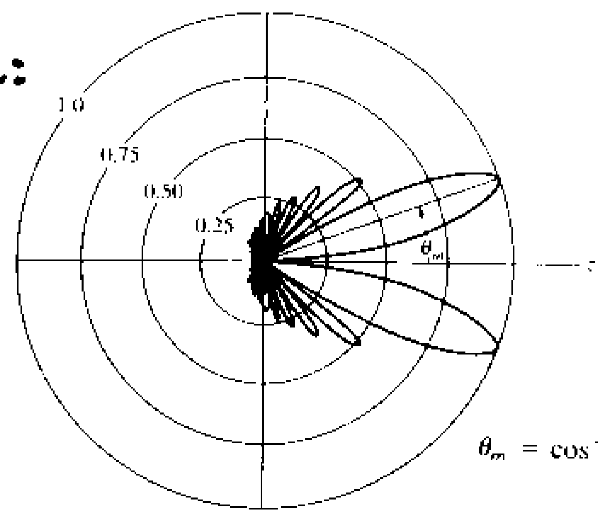
- Input impedance are traveling wave antennas are predominantly real. In this case, it is around 200 to 300 Ω.

Pattern: $F(\theta) = \sin\theta \dfrac{\sin[(\beta L/2)(1-\cos\theta)]}{\beta L/2(1-\cos\theta)}$

Example:

$L = 6\lambda$

$\theta_m = 20^\circ$



$\theta_m = \cos^{-1}\left(1 - \dfrac{0.371}{L/\lambda}\right)$

- For no reflection R_L is chosen to be 200 to 300 Ω.

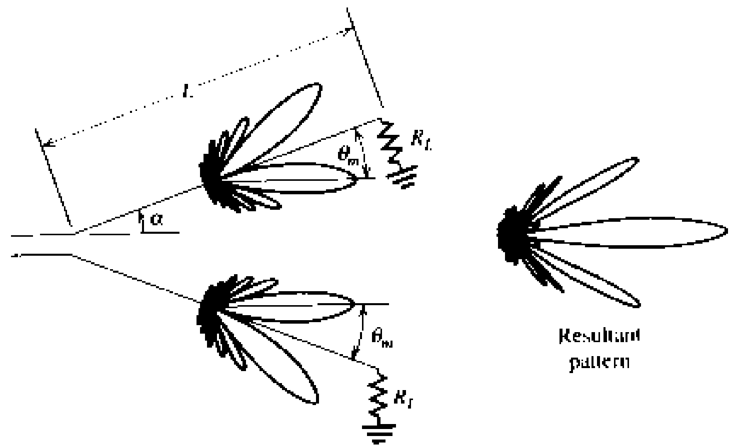
When this antenna is positioned near real earth, it is called "Beverage Antenna"

Broadband Antennas: Traveling Wave (3)

Long Vee antenna:

$L = 6\lambda$

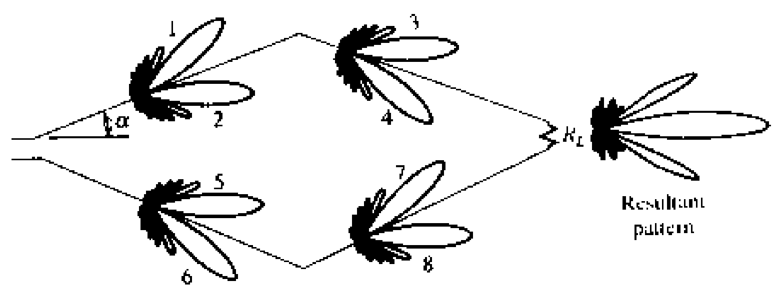
$\alpha = 16^\circ$



Rhombic antenna:

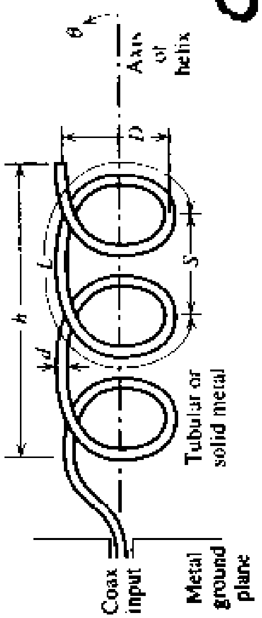
$L = 6\lambda$

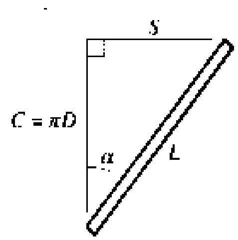
$\alpha = 16^\circ$



Broadband Antennas: Helical Antennas (1)

Geometry





Depending on the helix dimensions, this antenna can operate in "normal" or "axial" mode.

Parameters

D = diameter of helix (between centers of coil material)

C = circumference of helix = πD

S = spacing between turns = $C \tan \alpha$

α = pitch angle = $\tan^{-1} \dfrac{S}{C}$

L = length of one turn = $\sqrt{C^2 + S^2}$

N = number of turns

L_w = length of helix coil = NL

h = height = axial length = NS

d = diameter of helix conductor

Note:

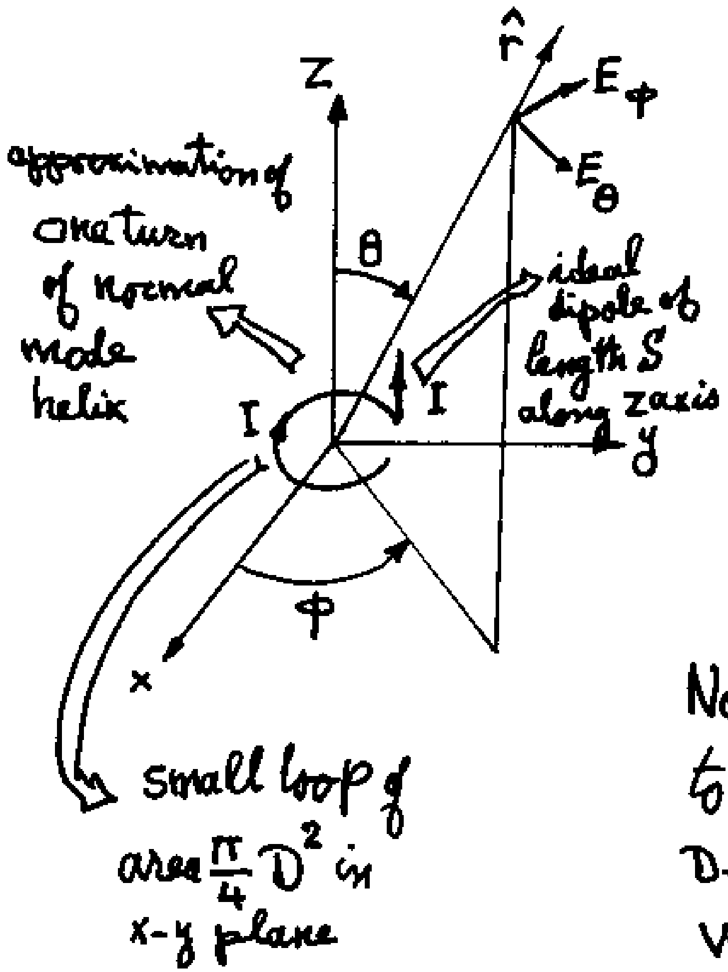
$S=0\ (\alpha=0°) \Rightarrow$ loop antenna

$D=0\ (\alpha=90°) \Rightarrow$ linear antenna

Broadband Antennas: Helical Antennas (2)

Normal Mode

For normal mode : $D \ll \lambda$; $L \ll \lambda$ $\Longrightarrow$ Radiation characteristics can be viewed as the combination of an ideal dipole and small loop



Ideal dipole : $\vec{E}_D = j\omega\mu I S \frac{e^{-j\beta r}}{4\pi r} \sin\theta \, \hat{\theta}$

Small loop : $\vec{E}_L = \eta \beta^2 \frac{\pi}{4} D^2 I \frac{e^{-j\beta r}}{4\pi r} \sin\theta \, \hat{\phi}$

Note that these fields are orthogonal to each other and 90° out of phase. Depending on their relative amplitudes various polarization can occur.

Broadband Antennas: Helical Antennas (3)

Normal Mode

$\eta = \sqrt{\frac{\mu}{\epsilon}}$

$\beta = \omega\sqrt{\mu\epsilon}$

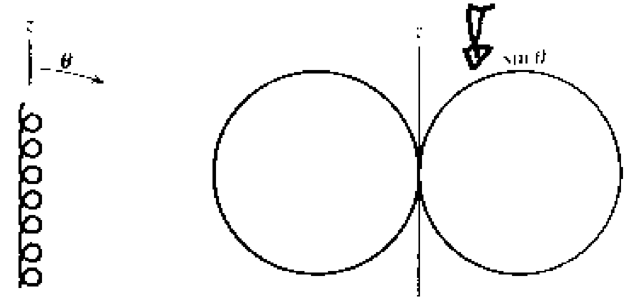
$\beta = \frac{2\pi}{\lambda}$

Polarization: $$\frac{|E_\theta|}{|E_\phi|} = \frac{\omega\mu I S}{\eta\beta^2 \frac{\pi}{4} D^2 I} = \frac{2S\lambda}{\pi^2 D^2} = \frac{2\frac{S}{\lambda}}{\left(\frac{\pi D}{\lambda}\right)^2}$$

or $$AR = \boxed{\frac{|E_\theta|}{|E_\phi|} = \frac{2\frac{S}{\lambda}}{\left(\frac{C}{\lambda}\right)^2}}$$

$C = \pi D =$ circumference

Radiation Pattern



In general, the polarization is elliptical.

Special cases:

$\boxed{S=0}$ $\Rightarrow$ small loop antenna $\Rightarrow$ horizontally polarized

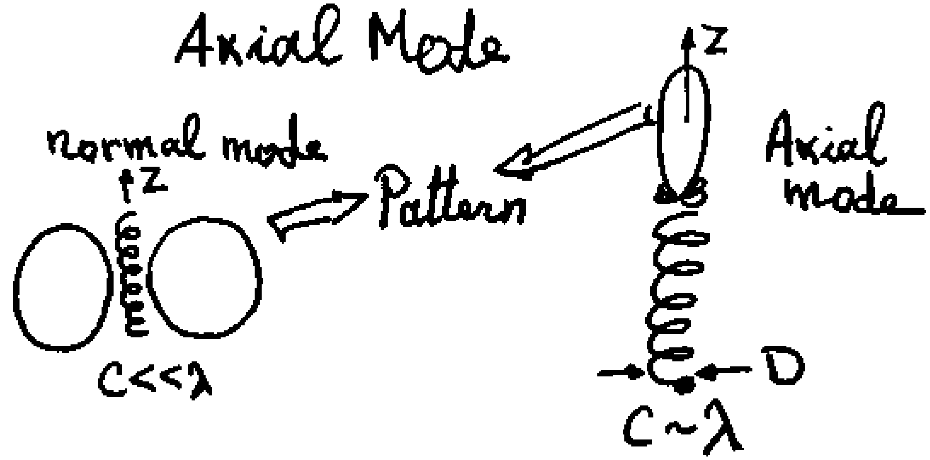
$\boxed{C=0}$ $\Rightarrow$ ideal dipole antenna $\Rightarrow$ vertically polarized

$\boxed{C = \pi D = \sqrt{2S\lambda}}$ $\Rightarrow$ circularly polarized ($\because |E_\theta| = |E_\phi|$)

note they were 90° out of phase

Broadband Antennas: Helical Antennas (4)

Axial Mode



Key features of axial mode:

$C = \pi D$

Axial mode helical antennas have very complex radiation mechanism. Kraus was the first to fully explain their characteristics.

Based on many experimental studies, it has been suggested that the helix carries a pure traveling wave that travels outward from the feed.

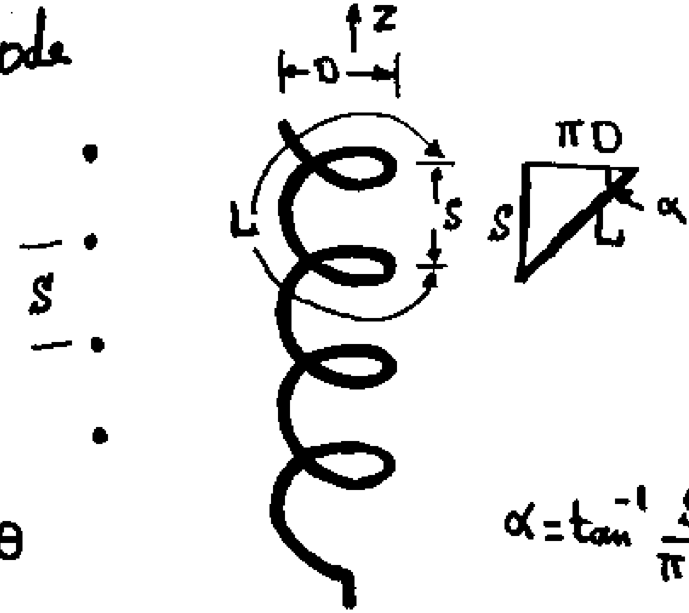
- operates well for

$$\frac{3}{4}\lambda \leq C \leq \frac{4}{3}\lambda$$

- $B_r = \frac{f_U}{f_L} = \frac{c/\lambda_U}{c/\lambda_L} = \frac{4/3}{3/4} = 1.78$ almost 2:1
- Almost circularly polarized
- 15 dB gain with small cross section

Broadband Antennas: Helical Antennas (5)
Axial Mode

The phase of the axial mode helix current shifts along the helix path.



$$\alpha = \tan^{-1}\frac{S}{\pi D}$$

Assuming that the element pattern of the non uniform current loop is $\cos\theta$ and applying the uniform array theory one obtains

normalization constant

No. of turns

$$F(\theta) = K\cos\theta\,\frac{\sin(N\psi/2)}{N\sin(\psi/2)}$$

$$\psi = \beta S\cos\theta + \alpha_h$$

Since the beam is along z-axis, i.e, $\theta = 0°$ and the peaks happens for $\psi = 0$, the initial thinking suggests that $\alpha_h = -\beta S$.

However, more is happening!

Broadband Antennas: Helical Antennas (6)
Axial Mode

What else is happening? Going around of $C=1\lambda$ loop additional 2π phase is required, then $\alpha_h = -\beta S - 2\pi$.

What else? However, since this antenna demostrate very small back lobe, it must operate near Hansen-Woodyard condition; therefore, a very realistic phase progression among the loops after each turn will be

$$\alpha_h = -(\underbrace{\beta S}_{\text{axial phase}} + \underbrace{2\pi}_{\text{phase due to } C=1\lambda \text{ turn}} + \underbrace{\frac{\pi}{N}}_{\text{Small back lobe condition}})$$

Broadband Antennas: Helical Antennas (7)
Axial Mode

Let us now assume that along the helical conductor the current travels with phase constant β_h. Then

$$\boxed{\beta_h \underbrace{L}_{\text{total length of one turn}} = \beta S + 2\pi + \frac{\pi}{N}} \Rightarrow \boxed{\beta_h = \frac{1}{L}\left(\beta S + 2\pi + \frac{\pi}{N}\right)}$$

Now, we can find current's "phase velocity" along the helix.

Phase velocity: $v = \dfrac{\omega}{\beta_h}$

speed of light: $c = \dfrac{\omega}{\beta}$

$$\Rightarrow \boxed{\underbrace{p}_{\text{velocity factor}} = \frac{v}{c} = \frac{\overbrace{\beta}^{2\pi/\lambda}}{\beta_h} = \frac{L/\lambda}{S/\lambda + (2N+1)/(2N)}}$$

For $p<1$ one encounters "slow wave".

Broadband Antennas: Helical Antennas (8)
Axial Mode

Based on the previous expressions the normalized pattern is:

$$F(\theta) = (-1)^{N+1} \sin\frac{\pi}{2N} \cos\theta \, \frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

$$\psi = \beta S(\cos\theta - 1) - 2\pi - \frac{\pi}{N}$$

For practical conditions of $\frac{3}{4} < C/\lambda < 4/3$; $12^\circ < \alpha < 15^\circ$, the following design parameters have been established

$$\boxed{HP = \frac{65^\circ}{\frac{C}{\lambda}\sqrt{N\frac{S}{\lambda}}}}$$

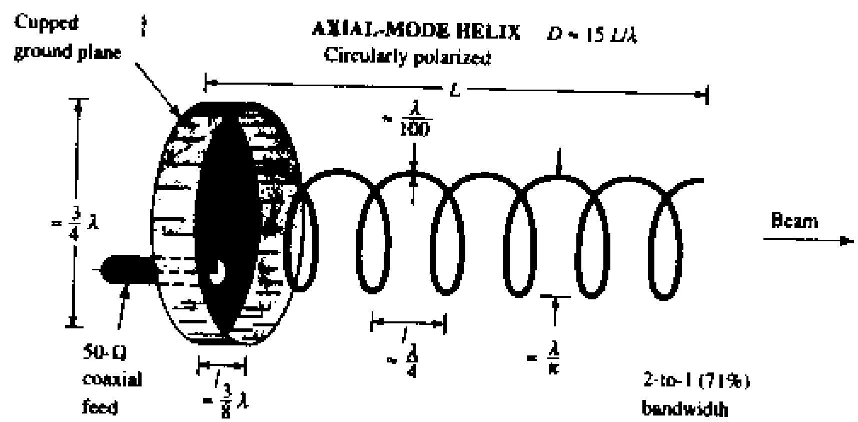
$$\boxed{G = \frac{26{,}000}{HP^2} = 6.2\left(\frac{C}{\lambda}\right)^2 N \frac{S}{\lambda}}$$

$$\boxed{B_r = 1.78}$$

$$\boxed{|AR| = \frac{2N+1}{N}}$$

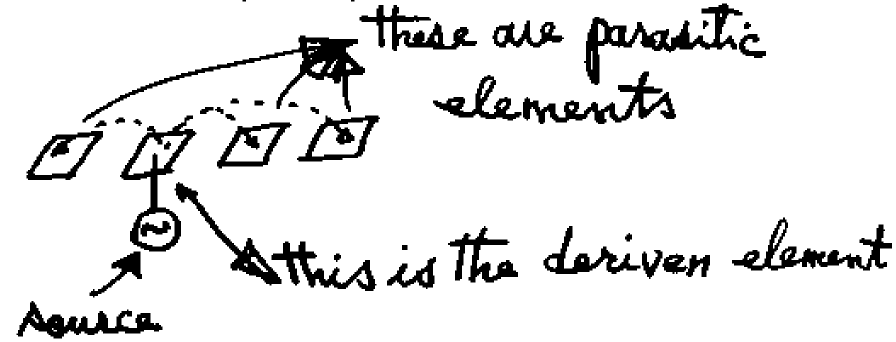
$$\boxed{R_A = 140\frac{C}{\lambda}\ \Omega}$$

An Axial Mode Helix invented by Kraus



Yagi-Uda Antennas (1)

Driven & Parasitic elements :



- Proper design of parasitic elements allows one to control the radiation characteristics of the antenna.

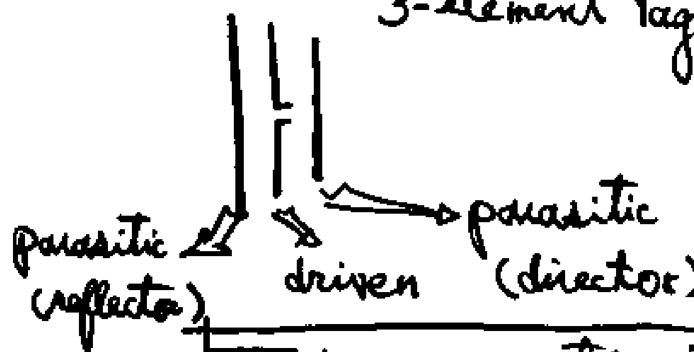
- Parasitic elements receive their excitation by near-field coupling from the driven element.

- A parasitic linear array of parallel dipoles is called a Yagi-Uda antenna, a Yagi-Uda array or simply "Yagi".

Yagi-Uda Antennas (2)

The basic unit of a Yagi antenna consists of three-elements. One driven and two parasitics.

3-element Yagi

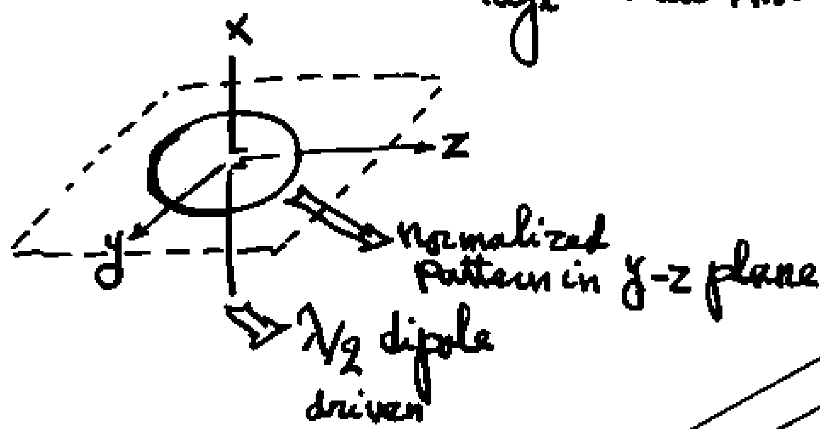


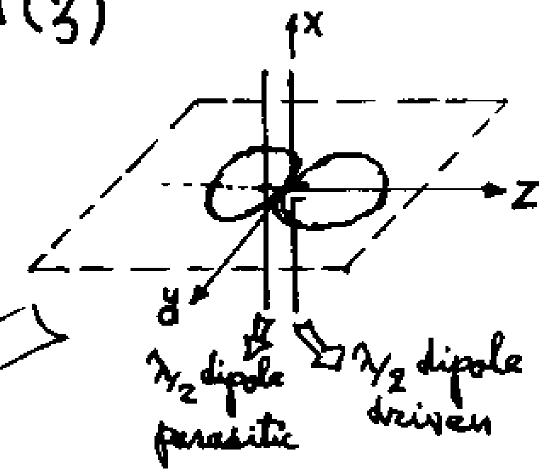
The design question is to find the length and spacing among elements.

History

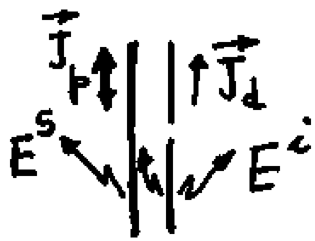
Shintaro Uda, an assistant prof. at Tokoku University in Japan conducted experiments on the use of parasitic elements. He wrote 11 articles in Japanese. The work of Uda was reviewerd in an article written in English by Uda's professor, H. Yagi in 1928.

Yagi- Uda Antennas (3)

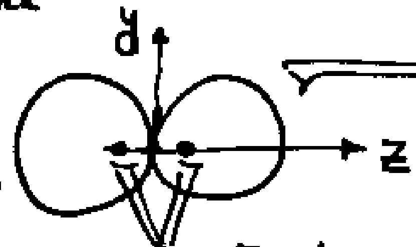




For closely spaced parasitic element, the current on the parasitic element is almost negative of the current on the driven element.



on parasitic element : $E^s + E^i = 0 \Longrightarrow I_p \approx -I_d$

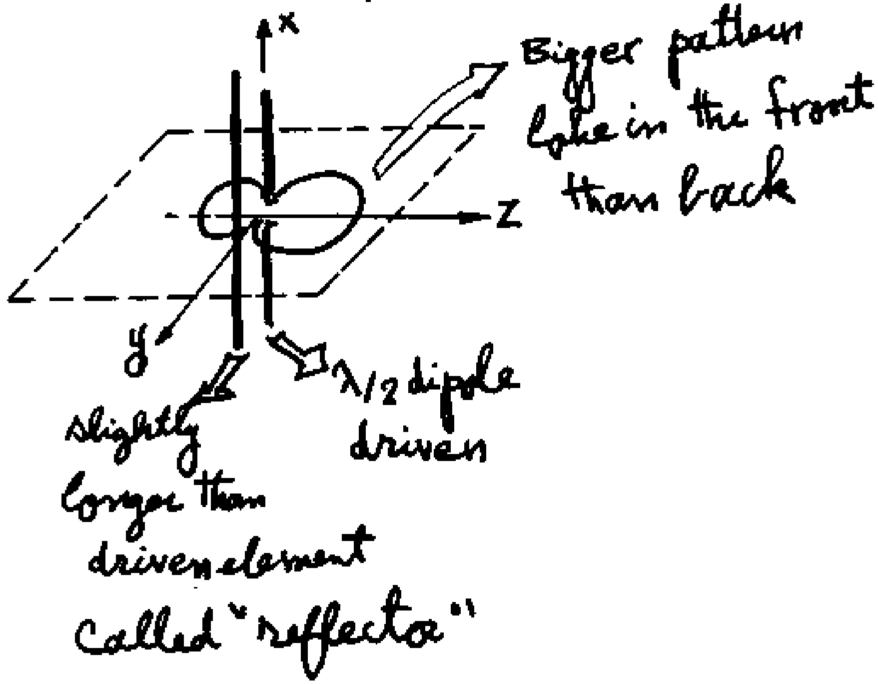


Similar lobes with null in y direction.

An end fire pattern results in y-z plane.

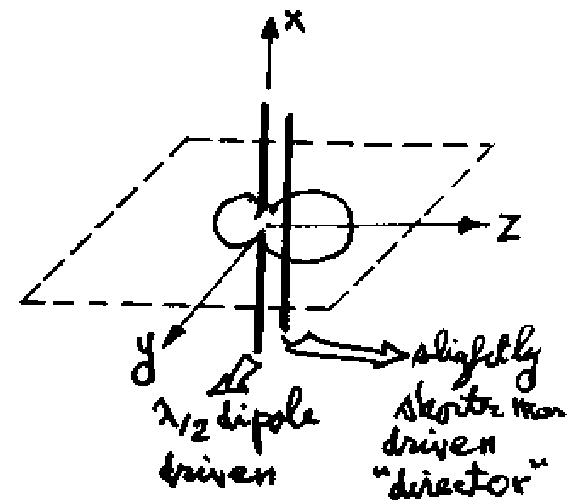
Yagi-Uda Antennas (4)

In order to create more directivity in the end fire direction, one should lengthen the parasitic element slighty longer than the driven. The parasitic element is now called "reflector".



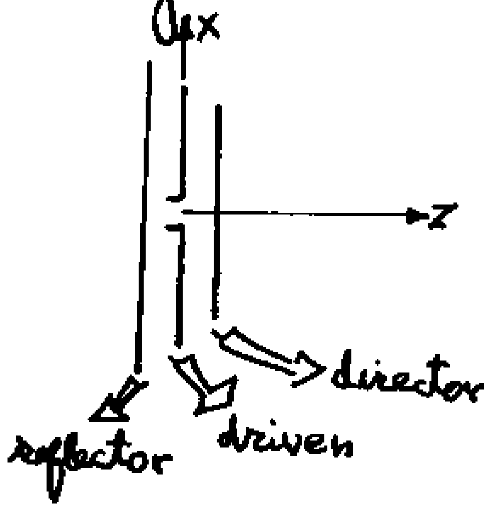
Director element:

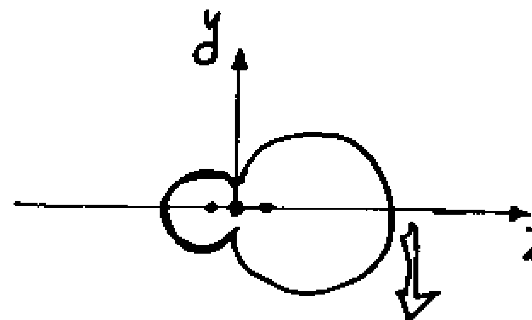
An element that is in front of driven and slightly shorter is called "director" element.



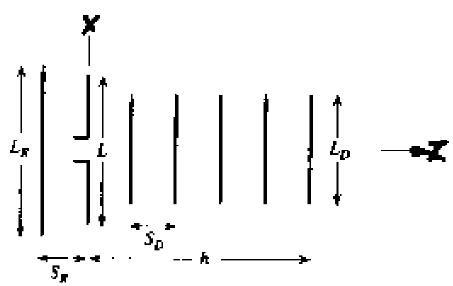
Yagi-Uda Antennas (5)

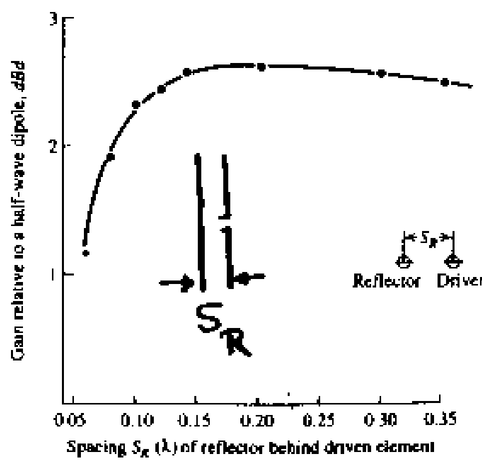
Putting all together:

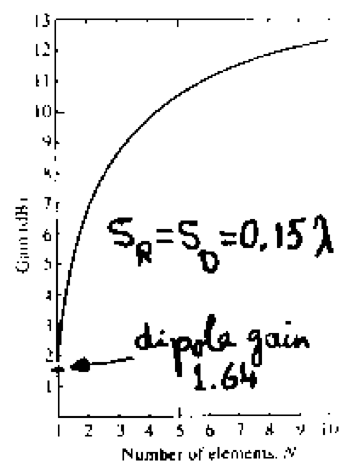




In order to further enhance the directivity of Yagi antenna, multi-director elements are typically used.

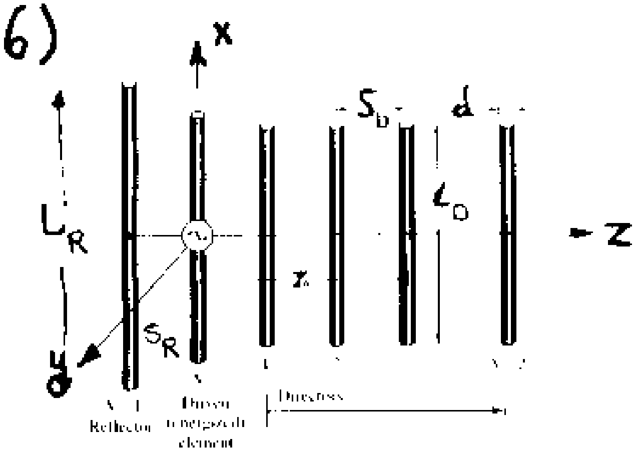






Yagi-Uda Antennas (6)

Many studies have been performed to optimize the Yagi-Uda antenna. Typically, method of moments is used to generate design curves and tables. One can then use these design curves and tables to construct the antenna.

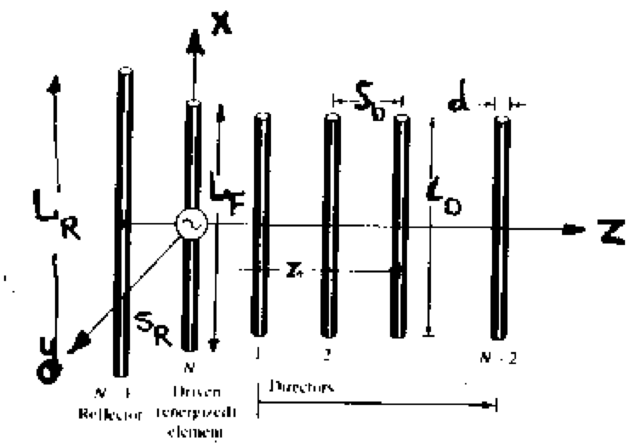


Optimized Lengths of Parasitic Dipoles for Yagi-Uda Array Antennas of Six Different Boom Lengths

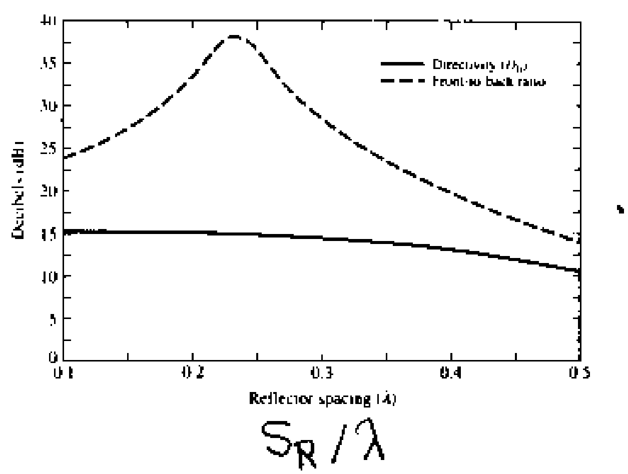
$d/\lambda = 0.0085$, $S_R = 0.2\lambda$	Boom length of Yagi-Uda Array, λ					
	0.4	0.8	1.20	2.2	3.2	4.2
Length of reflector, L_R/λ	0.482	0.482	0.482	0.482	0.482	0.475
Length of director D_n, (L_D/λ): D_1	0.442	0.428	0.428	0.432	0.428	0.424
D_2		0.424	0.420	0.415	0.420	0.424
D_3		0.428	0.420	0.407	0.407	0.420
D_4			0.428	0.398	0.398	0.407
D_5				0.390	0.394	0.403
D_6				0.390	0.390	0.398
D_7				0.390	0.386	0.394
D_8				0.390	0.386	0.390
D_9				0.398	0.386	0.390
D_{10}				0.407	0.386	0.390
D_{11}					0.386	0.390
D_{12}					0.386	0.390
D_{13}					0.386	0.390
D_{14}					0.386	
D_{15}					0.386	
Spacing between directors (S_D/λ)	0.20	0.20	0.25	0.20	0.20	0.308
Gain relative to half-wave dipole, dBd	7.1	9.2	10.2	12.25	13.4	14.2
Design curve (Fig. 5-37)	(A)	(C)	(C)	(B)	(C)	(D)
Front-to-back ratio, dB	8	15	19	23	22	20

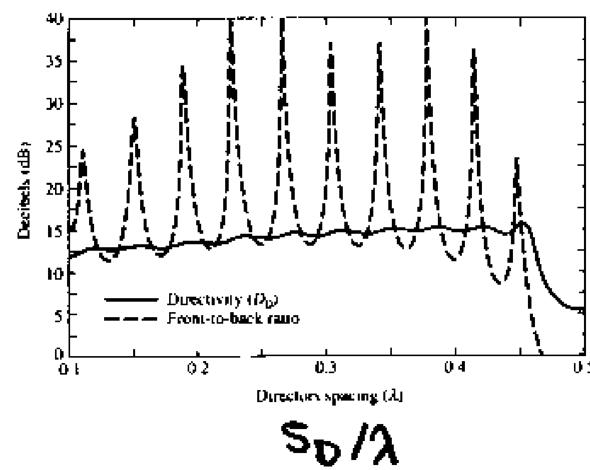
Source: P. P. Viezbicke, "Yagi Antenna Design," NBS Tech. Note 688, National Bureau of Standards, Washington, DC, Dec. 1968

Example: 15-element Yagi Uda Antenna



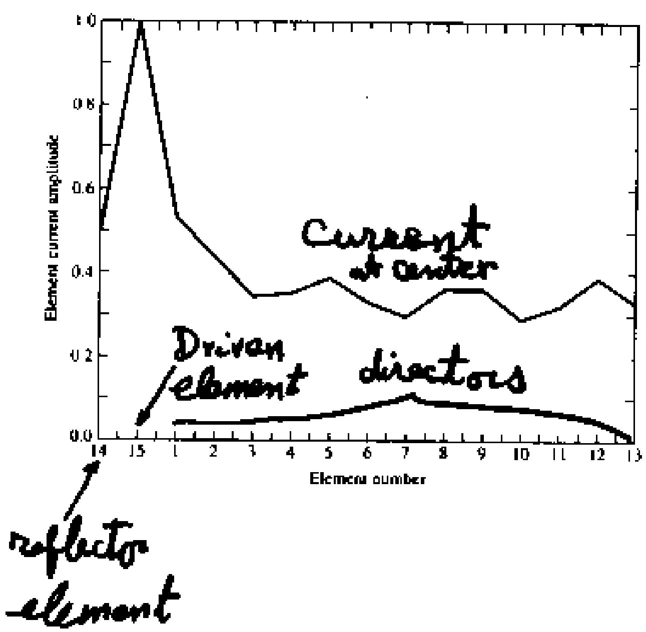
N = total number of elements = 15
number of directors = 13
number of reflectors = 1
number of exciters = 1
L_R = total length of reflector = 0.5λ
L_F = total length of feeder = 0.47λ
L_D = total length of each director = 0.406λ
S_R = spacing between reflector and feeder = 0.25λ
S_D = spacing between adjacent directors = 0.34λ
$d/2$ = a = radius of wires = 0.003λ

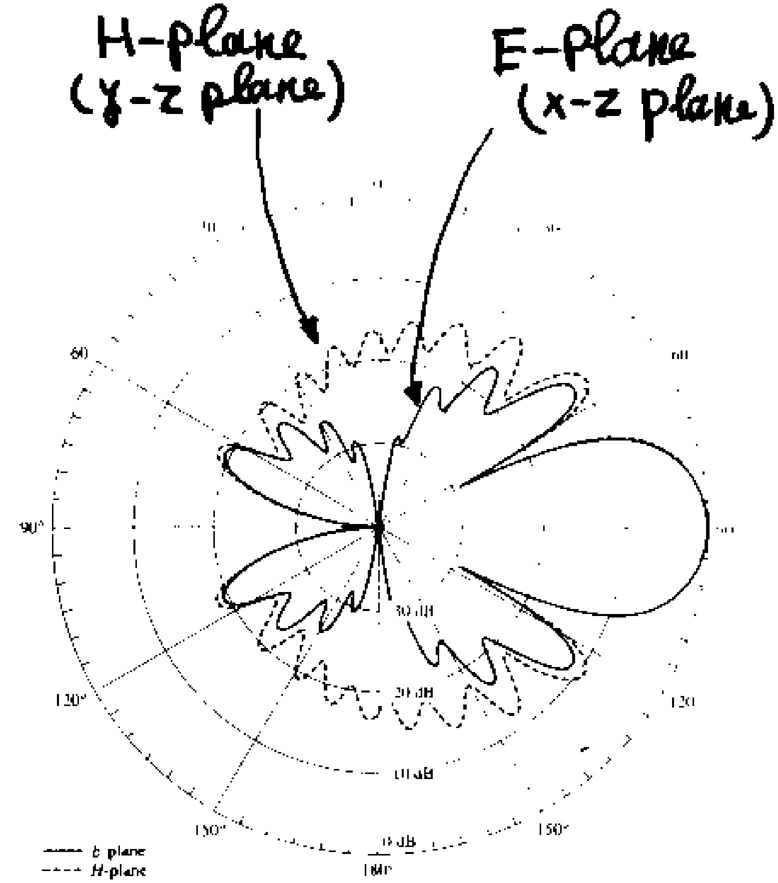




Example: 15-element Yagi-Uda Antenna

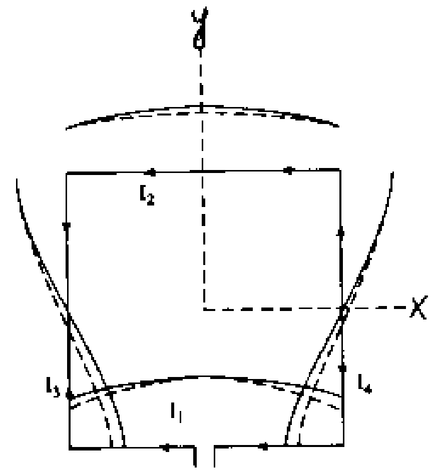
Yagi-Uda antennas are very popular and used a lot.





Resonant loop antennas: 1-λ square loop (1)

Loops can be made in many different shapes.



Simple Current model

$$\mathbf{I}_1 = \mathbf{I}_2 = -\hat{\mathbf{x}} I_o \cos(\beta x'), \qquad |x'| \le \frac{\lambda}{8}$$

$$\mathbf{I}_4 = -\mathbf{I}_3 = \hat{\mathbf{y}} I_o \sin(\beta y'), \qquad |y'| \le \frac{\lambda}{8}$$

Vector potential (Far field)

$$\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \int_{\text{loop}} \mathbf{I} e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'} \, dl$$

Source vector

$$\mathbf{r}_1' = x'\hat{\mathbf{x}} - \frac{\lambda}{8}\hat{\mathbf{y}} \qquad \mathbf{r}_2' = x'\hat{\mathbf{x}} + \frac{\lambda}{8}\hat{\mathbf{y}}$$

$$\mathbf{r}_3' = -\frac{\lambda}{8}\hat{\mathbf{x}} + y'\hat{\mathbf{y}} \qquad \mathbf{r}_4' = \frac{\lambda}{8}\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$$

Vector potential in details

$$\begin{aligned}
\mathbf{A} &= \mu \frac{e^{-j\beta r}}{4\pi r} I_o \Bigg[-\hat{\mathbf{x}} \int_{-\lambda/8}^{\lambda/8} \cos(\beta x') e^{j\beta x' \sin\theta\cos\phi} \left(e^{-j(\pi/4)\sin\theta\sin\phi} + e^{j(\pi/4)\sin\theta\sin\phi} \right) dx' \\
&\quad + \hat{\mathbf{y}} \int_{-\lambda/8}^{\lambda/8} \sin(\beta y') e^{j\beta y' \sin\theta\sin\phi} \left(-e^{-j(\pi/4)\sin\theta\cos\phi} + e^{j(\pi/4)\sin\theta\cos\phi} \right) dy' \Bigg] \\
&= \mu \frac{e^{-j\beta r}}{4\pi r} I_o \Bigg[-\hat{\mathbf{x}} 2 \cos\left(\frac{\pi}{4} \sin\theta \sin\phi \right) \int_{-\lambda/8}^{\lambda/8} \cos(\beta x') e^{j\beta x' \sin\theta\cos\phi} \, dx' \\
&\quad + \hat{\mathbf{y}} 2j \sin\left(\frac{\pi}{4} \sin\theta \cos\phi \right) \int_{-\lambda/8}^{\lambda/8} \sin(\beta y') e^{j\beta y' \sin\theta\sin\phi} \, dy' \Bigg]
\end{aligned}$$

The premiter of Resonant loop antennas are comparable to the wavelength.

Resonant loop antennas: 1-λ square loop (2)

Vector potential in a closed form:

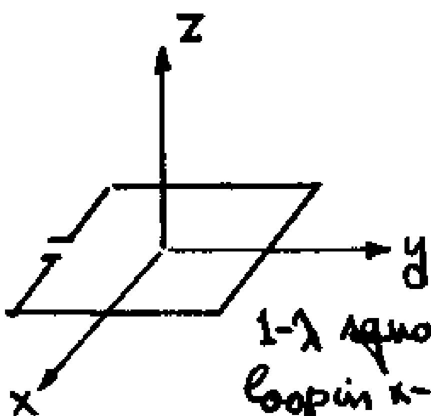
$$\cos\gamma = \sin\theta\cos\phi$$

$$\mathbf{A} = \mu\frac{e^{-j\beta r}}{4\pi r}\frac{2\sqrt{2}I_o}{\beta}\left\{\hat{\mathbf{x}}\,\frac{\cos[(\pi/4)\cos\Omega]}{\sin^2\gamma}\left[\cos\gamma\sin\left(\frac{\pi}{4}\cos\gamma\right) - \cos\left(\frac{\pi}{4}\cos\gamma\right)\right]\right.$$
$$\left. -\,\hat{\mathbf{y}}\,\frac{\sin[(\pi/4)\cos\gamma]}{\sin^2\Omega}\left[\cos\Omega\cos\left(\frac{\pi}{4}\cos\Omega\right) - \sin\left(\frac{\pi}{4}\cos\Omega\right)\right]\right\}$$

$$\cos\Omega = \sin\theta\sin\phi$$

Far-field expressions

$$E_\theta = -j\omega A_\theta = -j\omega\mathbf{A}\cdot\hat{\boldsymbol{\theta}} = -j\omega(A_x\hat{\mathbf{x}}\cdot\hat{\boldsymbol{\theta}} + A_y\hat{\mathbf{y}}\cdot\hat{\boldsymbol{\theta}})$$
$$= -j\omega(A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi)$$
$$E_\phi = -j\omega\mathbf{A}\cdot\hat{\boldsymbol{\phi}} = -j\omega(-A_x\sin\phi + A_y\cos\phi)$$

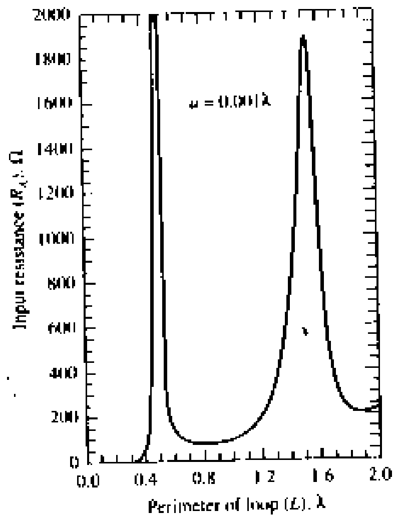


1-λ square loop in x-y plane

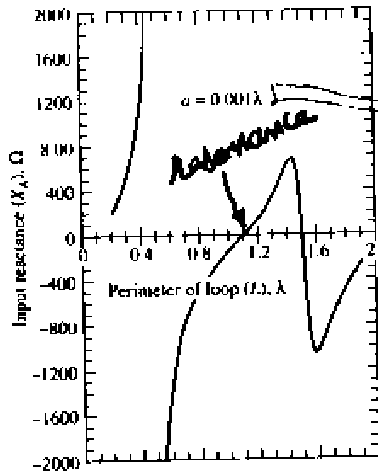
$$E_\theta = \frac{jI_o\eta e^{-j\beta r}}{\sqrt{2}\pi r}\cos\theta\left\{\frac{\sin\phi\sin[(\pi/4)\sin\theta\cos\phi]}{1-\sin^2\theta\sin^2\phi}\right.$$
$$\cdot\left[\sin\theta\sin\phi\cos\left(\frac{\pi}{4}\sin\theta\sin\phi\right) - \sin\left(\frac{\pi}{4}\sin\theta\sin\phi\right)\right]$$
$$-\frac{\cos\phi\cos[(\pi/4)\sin\theta\sin\phi]}{1-\sin^2\theta\cos^2\phi}$$
$$\left.\cdot\left[\sin\theta\cos\phi\sin\left(\frac{\pi}{4}\sin\theta\cos\phi\right) - \cos\left(\frac{\pi}{4}\sin\theta\cos\phi\right)\right]\right\}$$

$$E_\phi = \frac{jI_o\eta e^{-j\beta r}}{\sqrt{2}\pi r}\left\{\frac{\cos\phi\sin[(\pi/4)\sin\theta\cos\phi]}{1-\sin^2\theta\sin^2\phi}\right.$$
$$\cdot\left[\sin\theta\sin\phi\cos\left(\frac{\pi}{4}\sin\theta\sin\phi\right) - \sin\left(\frac{\pi}{4}\sin\theta\sin\phi\right)\right]$$
$$+\frac{\sin\phi\cos[(\pi/4)\sin\theta\sin\phi]}{1-\sin^2\theta\cos^2\phi}$$
$$\left.\cdot\left[\sin\theta\cos\phi\sin\left(\frac{\pi}{4}\sin\theta\cos\phi\right) - \cos\left(\frac{\pi}{4}\sin\theta\cos\phi\right)\right]\right\}$$

Resonant loop antennas: 1-λ square loop (3)



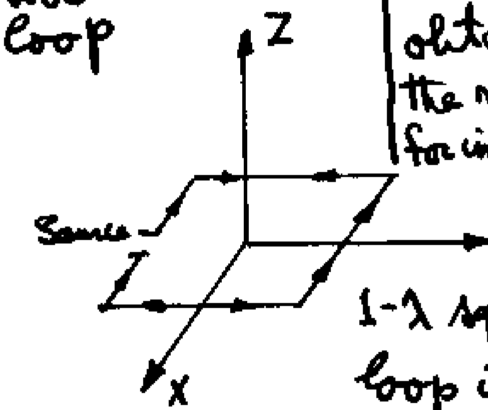
Input resistance



Input reactance

radius of wire loop

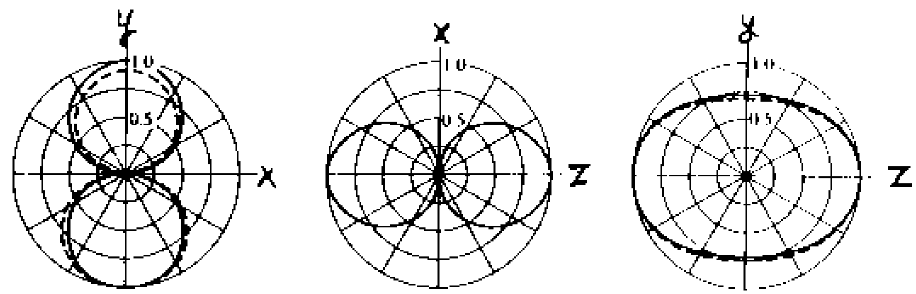
MoM is used to obtain the results for impedance



1-λ square loop in x-y plane

Directivity 1-λ loop $= 3.09$ dB

Note: The patterns of resonant loop antenna is very different than small loop.



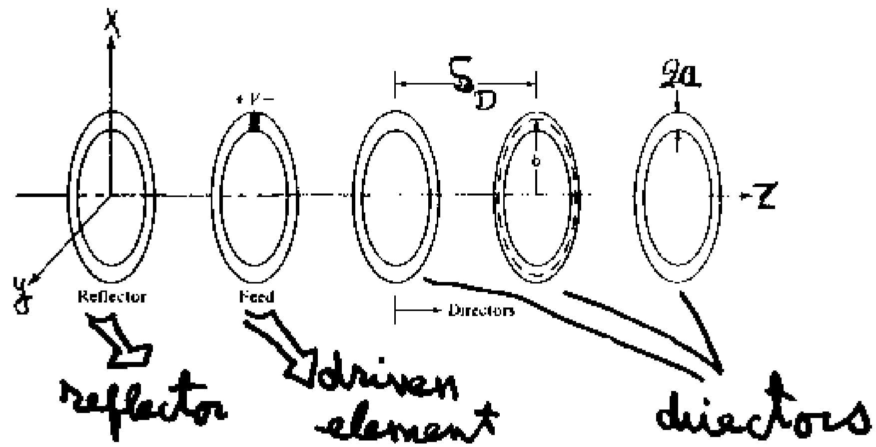
(a) The xy-plane (the plane of the loop and an E-plane) normalized pattern plot of E_ϕ. In this plane, HP = 94°.

(b) The xz plane (an E plane) normalized pattern plot of E_ϕ. In this plane, HP = 85°. The patterns from the two methods coincide in this plane.

(c) The yz plane (the H-plane) pattern plot of E_θ.

Yagi-Uda loop antennas

Loops are near resonant dimensions.



Observations: there exists tremendous amount of "clever" antenna designs for variety of applications.

The Big Question?

How can one design a low profile antenna near a P.e.c. ground plane that radiates effectively?

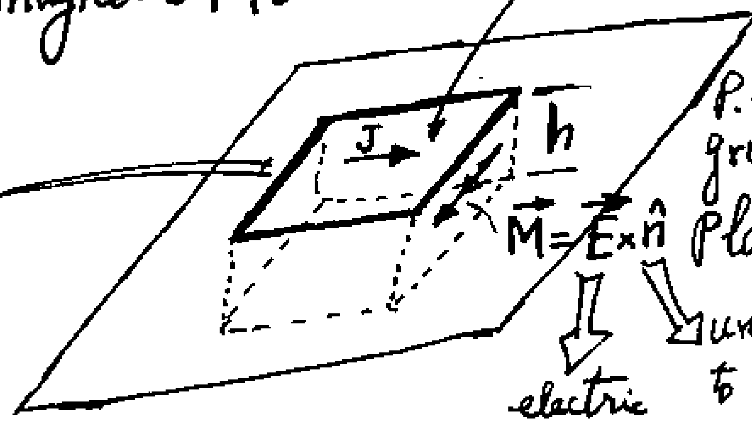
Options

	Low Profile	effective radiation	Comment
$\vec{J}$ Vertical electric antenna, P.e.c.	☹	☺	Cannot be made very low profile
$\vec{J}$ horizontal electric antenna, P.e.c.	☺	☹	poor radiation when it is low profile
$\vec{M}$ Vertical magnetic antenna, P.e.c.	☹	☹	No good
$\vec{M}$ Horizontal magnetic antenna, P.e.c.	☺	☺	Challenge is how to create horizontal magnetic current

The Big Idea!

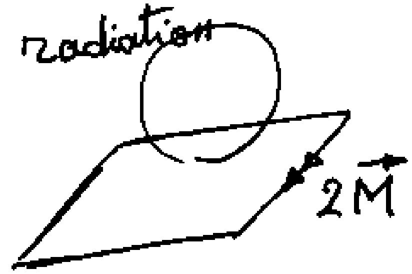
Our understanding so far tells us that there is no "real" magnetic currents. However, the equivalence theorem may allow us to interpret "$\vec{E}\times\hat{n}$" as magnetic $\vec{M}$.

Note: One gets very poor radiation from horizontal $\vec{J}$ when $h/\lambda \ll 1$



Note: Since $\vec{E}$ is vertically oriented, $\vec{M}$ is horizontally oriented.

When h is small, i.e., low profile antenna, the image theory gives an effective loop of magnetic current.



Microstrip Antennas (0)

- The fundamental idea of the microstrip antennas is to somehow establish the "magnetic current" in the horizontal direction which can radiate in the presence of a very nearby ground plane.

- This simple idea has revolutionized the technology and applications of the low profile and conformal antennas.

Microstrip Antennas (1)

Modern applications of antennas have necessitated to employ antennas that are conformal and do not stick out. The concept of "printed antennas" was conceived in early 1930's. Since early 1970's this concept has been one of the major areas for developments of new antennas and research activities. There are 100's of papers and books written in this area and many powerful simulation tools have been developed.

Microstrip Antennas (2)

What is the big challenge? "The tradeoff in microstrip antennas is to design a patch with loosely bound fields extending into space while keeping the fields tightly bound to the feeding circuitry."

Observation: "The substrate requirements for circuit operation are typically inconsistent with those needed for antenna operation. It is not possible to realize an efficient antenna and a nonradiating circuit upon the same substrate."

Microstrip Antennas (3)

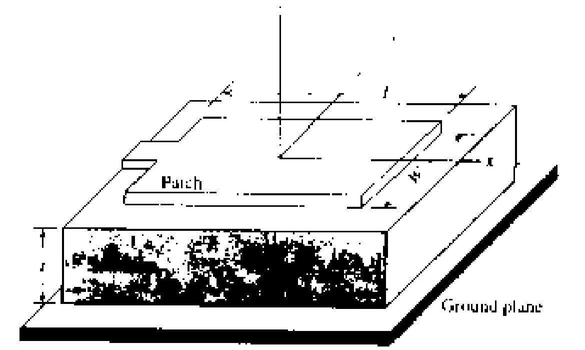
Common features of microstrip antennas:

(i) A very thin flat metallic region often called the patch

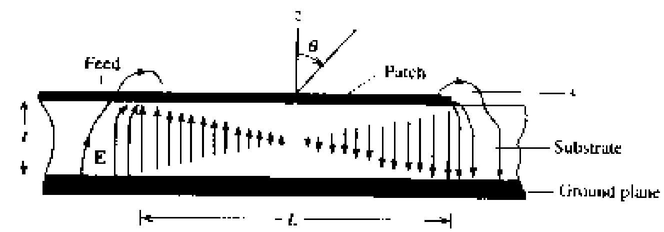
(ii) A dielectric substrate

(iii) A ground plane

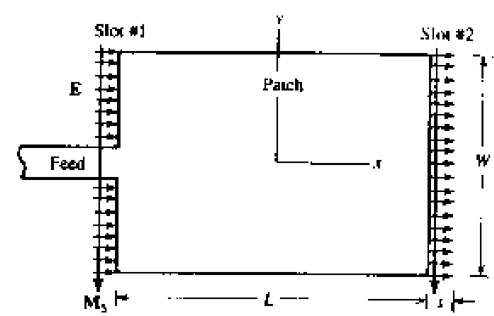
(iv) A feed, which supplies the rf power.



(a) Geometry for analyzing the edge-fed microstrip patch antenna



(b) Side view showing the electric fields



Microstrip antennas are often made by etching the patch from a printed board clad with conductor on both sides.

Applications

Typical applications for printed-antenna technology

Aircraft antennas	Communication and navigation Altimeters Blind-landing systems
Missiles and telemetry	Stick-on sensors Proximity fuzes Millimetre devices
Missile guidance	Seeker monopulse arrays Integral radome arrays
Adaptive arrays	Multi-target acquisition Semiconductor integrated array
Battlefield communications and surveillance	Flush-mounted on vehicles
SATCOMS	Domestic DBS receiver Vehicle-based antenna Switched-beam arrays
Mobile radio	Pagers and hand telephones Manpack systems
Reflector feeds	Beam switching
Remote Sensing	Large lightweight apertures
Biomedical	Applicators in microwave cancer therapy
Covert antennas	Intruder alarms Personal communication

Various Ways to feed Microstrip Patch

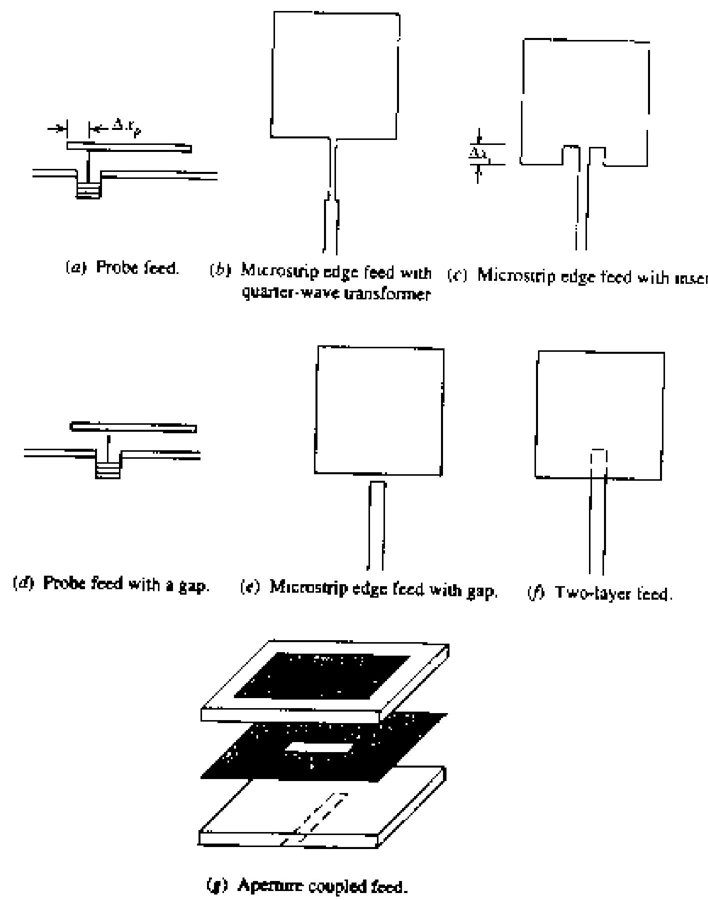
There are three major feeding mechanisms:

(i) Probe fed

(ii) Strip line fed

(iii) Aperture coupled fed

Depending on applications and cost of implementation each mechanism has advantages and disadvantages.



(a) Probe feed. (b) Microstrip edge feed with quarter-wave transformer (c) Microstrip edge feed with inset.

(d) Probe feed with a gap. (e) Microstrip edge feed with gap. (f) Two-layer feed.

(g) Aperture coupled feed.

Properties

Some commonly acknowledged properties of microstrip antennas

Advantages	Disadvantages
Thin profile	Low efficiency
Light weight	Small bandwidth
Simple to manufacture	Extraneous radiation from feeds, junctions and surface waves
Can be made conformal	Tolerance problems
Low cost	Require quality substrate and good temperature tolerance
Can be integrated with circuits	High-performance arrays require complex feed systems
Simple arrays readily created	Polarisation purity difficult to achieve

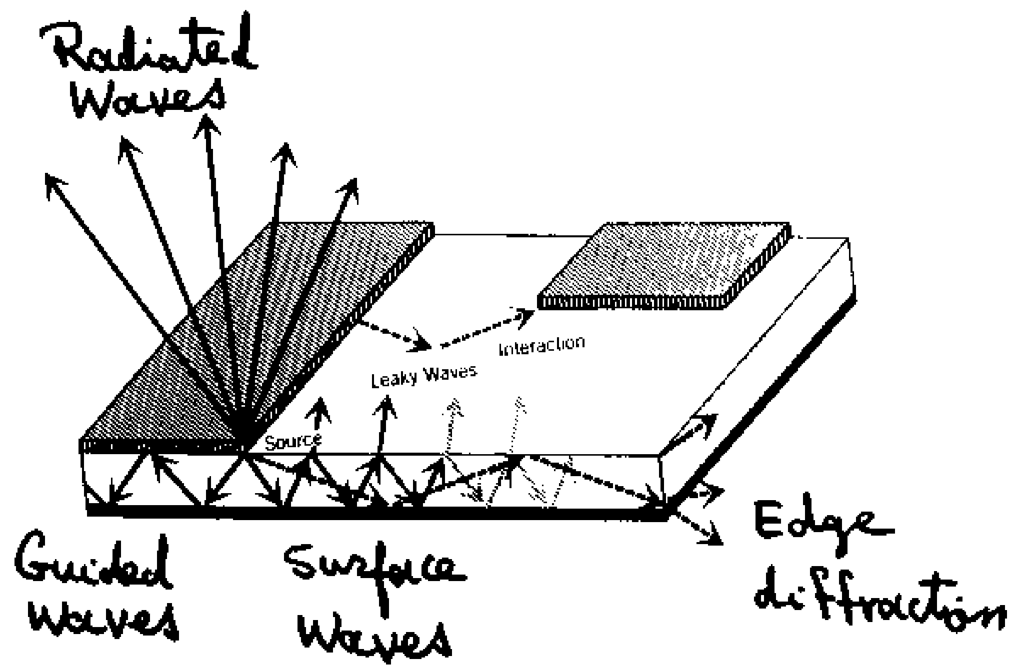
Substrates

Representative substrate list

ε_r	Material
1·0	Aeroweb (honeycomb)
1·06	Eccofoam PP-4 (flexible low-loss plastic foam sheet)
1·4	Thermoset microwave foam material
2·1	RT Duroid 5880 (microfiber Teflon glass laminate)
2·32	Polyguide 165 (polyolefin)
2·52	Fluorglas 600/1 (PTFE impregnated glass cloth)

ε_r	Material
3·75	Quartz (fuzed silica)
6·0	RT Duroid 6006 (ceramic-loaded PTFE)
9·9	Alumina
2·62	Rexolite 200 (cross-linked styrene copolymer)
3·20	Schaefer Dielectric Material, PT (polystyrene with titania filler)
3·5	Kapton film (copper clad)

Radiation Mechanisms in Microstrip Antennas



In general, complex radiation mechnisms affect various features of the antenna.

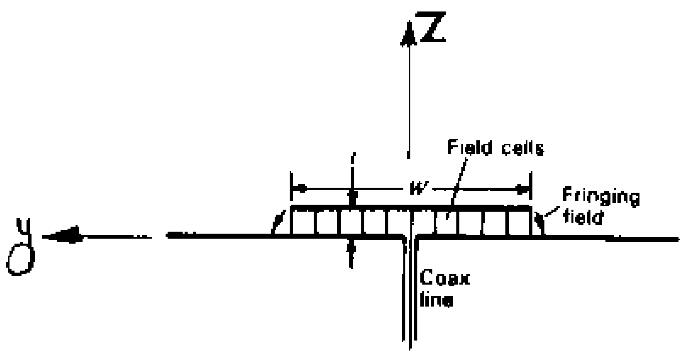
Microstrip Analysis Methodology

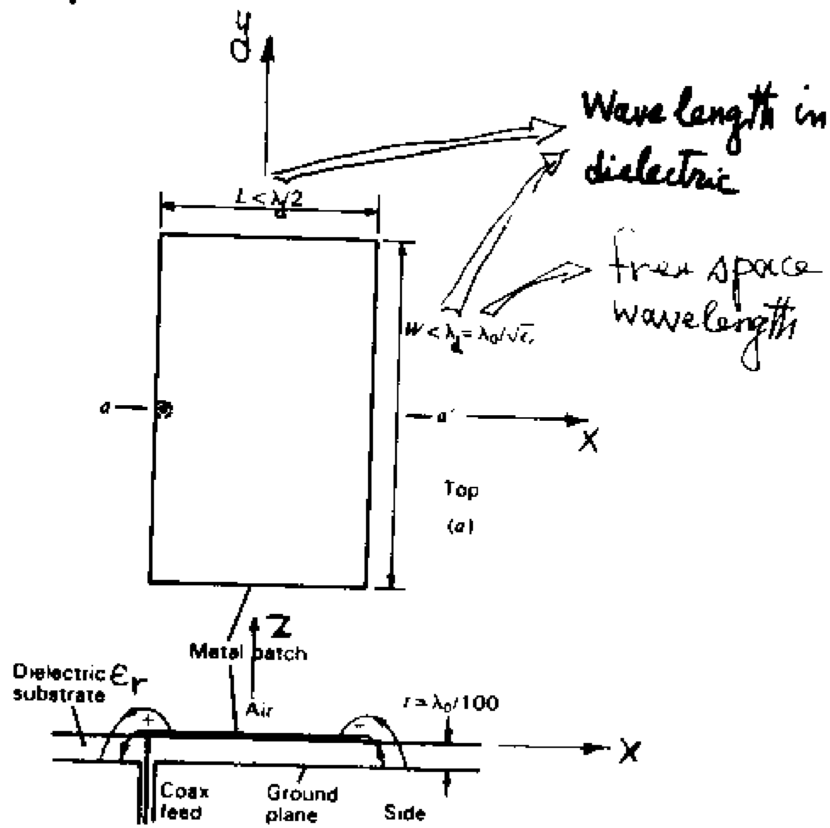
- Slot aperture model
- Cavity model
- Full wave solutions
  - Integral eq. & MoM
  - Finite Difference Time Domain
  - Finite Element Method
- Hybrid Techniques

For simple structure the first two methods provide useful physical insight.

Simple rectangular patch antenna

Rectangular patch antenna is one of the most commonly used microstrip antennas.





Note: $\lambda_d = \dfrac{\lambda_0}{\sqrt{\epsilon_r}}$

Wavelength in dielectric substrate is "shorter" than the wavelength in free space.

Some useful formulas for Patch antenna Design

Length:

$$L = 0.49\lambda_d = 0.49 \frac{\lambda}{\sqrt{\epsilon_r}}$$ half-wave Patch

Patterns:

E-plane (x-z):

$$F_E(\theta) = \cos\left(\frac{\beta L}{2}\sin\theta\right), \quad \beta = \frac{2\pi}{\lambda} \text{ (free space)}$$

H-plane (y-z):

$$F_H(\theta) = \cos\theta \frac{\sin\left[\frac{\beta W}{2}\sin\theta\right]}{\frac{\beta W}{2}\sin\theta}$$

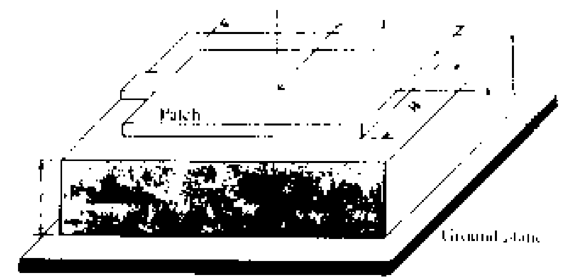
Impedance at resonant condition:

$$\boxed{Z_A = 90 \frac{\epsilon_r^2}{\epsilon_r - 1}\left(\frac{L}{W}\right)^2 \text{ ohms}} \quad \epsilon_r > 2$$

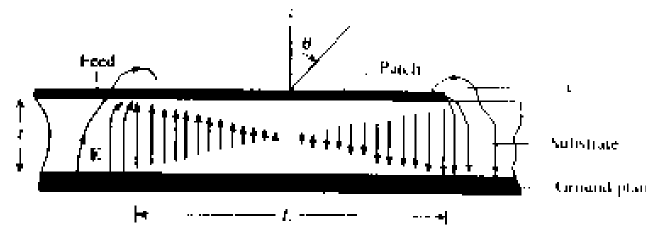
Bandwidth:

$$\boxed{B = 3.77 \frac{\epsilon_r - 1}{\epsilon_r^2} \frac{W}{L} \frac{t}{\lambda}}$$

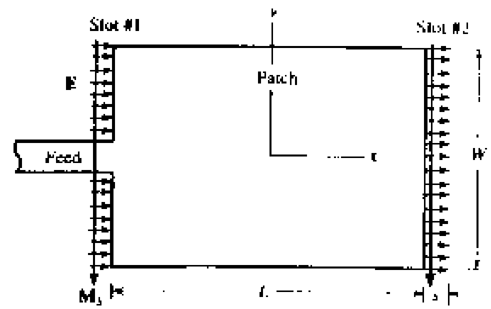
VSWR 2:1



(a) Geometry for analyzing the edge fed microstrip patch antenna

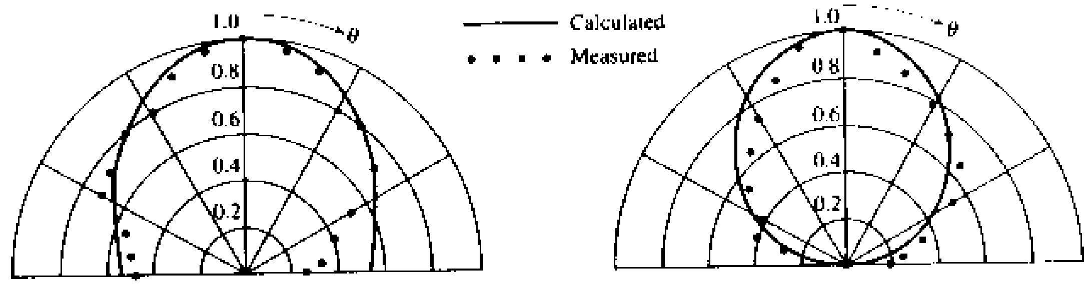


(b) Side view showing the electric fields.

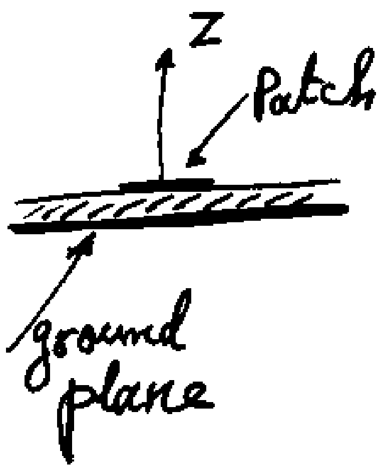


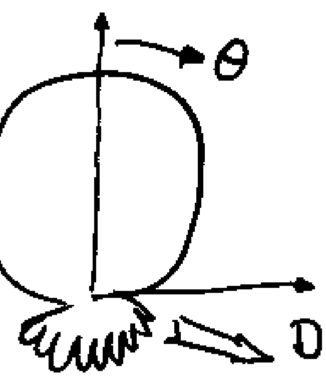
Note: There exists many sophisticated computer programs for accurate and detailed design of microstrip antennas.

A typical Pattern for a Rectangular Microstrip Antenna



E-plane H-plane





Depending on the size of the ground plane some radiation is observed under the ground plane

Microstrip Antennas

Factors constraining the bandwidth of microstrip antenna elements and arrays

Element	Array
Input-impedance characteristic	Surface-waves
Side-lobe level	Element mutual coupling
Cross-polarisation level	Feeder radiation
Circular polarisation (axial ratio)	Corporate feed and mismatch
Pattern shape (E- and H-plane symmetry)	Scanning loss
Element gain	
Efficiency	
Feeder radition	

A Simple Design Procedure for Microstrip Antennas (1)

- There are available simple formulas to allow antenna designers to initiate a reasonably good microstrip antenna.

Problem statement: Given substrate's relative permittivity "ϵ_r" and thickness "t" and the desired resonant frequency "f_r" find the length "L" and width "W" of the rectangular patch antenna.

Solution: There are essentially four steps as summarized next.

A Simple Design Procedure for Microstrip Antennas (2)

Step 1: For an efficient radiation, a practical width is

speed of light in free space

$$W = \frac{1}{2 f_r \sqrt{\mu_0 \epsilon_0}} \sqrt{\frac{2}{\epsilon_r + 1}} = \frac{c}{2 f_r} \sqrt{\frac{2}{\epsilon_r + 1}}$$

frequency in Hz

or

$$\boxed{W^{cm} = \frac{30}{2 f_r(\text{GHz})} \sqrt{\frac{2}{\epsilon_r + 1}} = \frac{\lambda^{cm}}{2} \sqrt{\frac{2}{\epsilon_r + 1}}}$$

free space wavelength

Note: $\lambda^{cm} \approx \dfrac{30}{f_r\,(\text{in GHz})}$

Step 2: Determine the effective dielectric constant

thickness of the substrate

$$\epsilon_{eff} = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \left[1 + 12 \frac{t}{W}\right]^{-1/2}$$

$$\sqrt{1 + 12 \frac{t}{W}}$$

A Simple Design Procedure for Microstrip Antennas (3)

Step 3: Determine the length extension ΔL on each side

$$\frac{\Delta L}{t} = 0.412\,\frac{(\epsilon_{eff}+0.3)\left(\frac{W}{t}+0.264\right)}{(\epsilon_{eff}-0.258)\left(\frac{W}{t}+0.8\right)}$$

Step 4: The actual length is

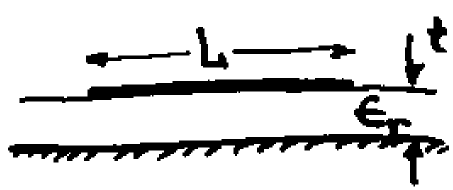
wavelength in dielectric

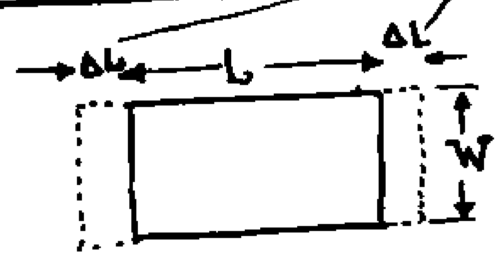
$$\boxed{L = \frac{\lambda_d}{2} - 2\Delta L = \frac{\lambda}{2\sqrt{\epsilon_r}} - 2\Delta L}$$

$$\lambda^{cm} = \frac{30}{f_r(\mathrm{GHz})}$$

free space wavelength

Geometry





due to fringing effects

Approximate performance trade-offs for a rectangular patch

Requirement	Substrate height	Substrate relative permittivity	Patch width
High radiation efficiency	thick	low	wide
Low dielectric loss	thin	low	—
Low conductor loss	thick	—	—
Wide (impedance) bandwidth	thick	low	wide
Low extraneous (surface wave) radiation	thin	low	—
Low cross polarisation	—	low	—
Light weight	thin	low	—
Strong	thick	high	—
Low sensitivity to tolerances	thick	low	wide

Approximate performance trade-offs for an array of circular patches

Requirement	Substrate height	Substrate relative permittivity
High efficiency	thick	low
Low feed radiation	thin	high
Wide (impedance) bandwith	thick	low
Low extraneous surface-wave radiation	thin	low
Low mutual coupling	thick	low
Low sensitivity to tolerances	thick	low

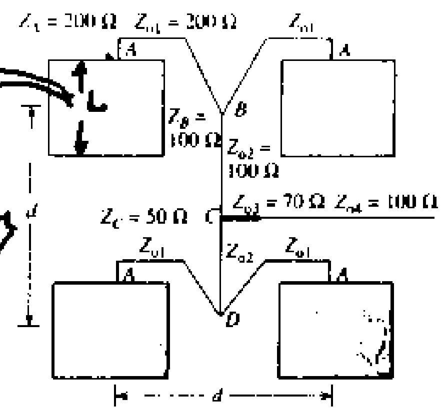
Microstrip Antenna Arrays

Microstrip antenna arrays are often used in many modern applications. They can be designed in various configurations.

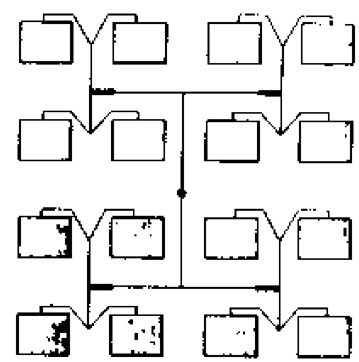
Key observations:

- Patch dimensions are controled by the wavelength in the dielectric $\lambda_d = \frac{\lambda}{\sqrt{\epsilon_r}}$
- Array element spacing is controled by the free space wavelength λ.

Example:



4-element subarray



16-element array

Fundamental issues that will continue to be addressed
Bandwidth extension techniques
Control of radiation patterns involving sidelobes, beamshaping, cross-polarisation, circular polarisation, surface-wave and ground-plane effects
Reducing loss and increasing radiation efficiency
Optimal feeder systems (array architecture)
Improved lower-cost substrates and radomes
Tolerance control and operational factors

Faraday's Law & Stokes Theorem

Faraday's Law : $\displaystyle\oint_C E \cdot d\ell = -\frac{d}{dt}\int_S B \cdot ds$

Stokes Theorem: $\displaystyle\oint_C E \cdot d\ell = \int_S (\nabla \times E) \cdot ds$

True for any C & S

One of Maxwell's Eq. : $\boxed{\nabla \times E = -\dfrac{\partial B}{\partial t}}$

Dimensions : L^{-1} $\quad MLT^{-2}Q^{-1}$ $\quad T^{-1}$ $\quad MT^{-1}Q^{-1}$

Ampere's Law & Stokes Theorem

Ampere's Law : $\displaystyle\oint_C H \cdot d\ell = \int_S J \cdot ds + \frac{d}{dt}\int_S D \cdot ds$

Stokes Theorem: $\displaystyle\oint_C H \cdot d\ell = \int_S (\nabla \times H) \cdot ds$

True for any C & S

One of Maxwell's Eq. : $\boxed{\nabla \times H = J + \dfrac{\partial D}{\partial t}}$

Dimensions : L^{-1} $\quad L^{-1}T^{-1}Q$ $\quad L^{-2}T^{-1}Q$ $\quad T^{-1}$ $\quad L^{-2}Q$

Gauss' Law & Divergence Theorem

Gauss' Law: $\oint_S D \cdot ds = \int_V \rho \, dv$

Divergence Theorem: $\oint_S D \cdot ds = \int_V \nabla \cdot D \, dv$

True for any S & V

$\Downarrow$

One of Maxwell's Eq.: $\boxed{\nabla \cdot D = \rho}$

Dimensions: L^{-1} $\quad L^{-2}Q \quad L^{-3}Q$

Gauss' Law & Divergence Theorem

Gauss' Law: $\oint_S B \cdot ds = 0 = \int_V 0 \, dv$

Divergence Theorem: $\oint_S B \cdot ds = \int_V D \cdot B \, dv$

True for any S & V

$\Downarrow$

One of Maxwell's Eq.: $\boxed{\nabla \cdot B = 0}$

Dimensions: $L^{-1} \quad MT^{-1}Q^{-1}$

Conservation of Charge & Divergence Theorem

Conservation of Charge : $\oint_S \mathbf{J}\cdot d\mathbf{s} = -\dfrac{d}{dt}\displaystyle\int_V \rho\, dv$

Divergence Theorem : $\oint_S \mathbf{J}\cdot d\mathbf{s} = \displaystyle\int_V \nabla\cdot \mathbf{J}\, dv$

True for any S & V

$\Downarrow$

Continuity Eq. : $\boxed{\nabla\cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}}$

Dimensions : L^{-1} $\quad L^{-2}T^{-1}Q$ $\quad T^{-1}$ $\quad L^{-3}Q$

Maxwell's Equations in Integral and Differential Forms and Boundary Conditions at an Interface

Integral Form	Differential Form	Boundary Condition
$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\dfrac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$	$\nabla\times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$E_{t1} = E_{t2}$ or $\mathbf{n}\times(\mathbf{E}_1 - \mathbf{E}_2) = 0$
$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{s} + \dfrac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{s}$	$\nabla\times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$H_{t1} - H_{t2} = J_{s(n)}$ or $\mathbf{n}\times(\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$ σ_1, σ_2 finite: $H_{t1} = H_{t2}$ $\sigma_2 \to \infty$: $\mathbf{n}\times \mathbf{H}_1 = \mathbf{J}_s$
$\oint_S \mathbf{D}\cdot d\mathbf{s} = \int_V \rho\, dv$	$\nabla\cdot \mathbf{D} = \rho$	$D_{n1} - D_{n2} = \rho_s$ or $\mathbf{n}\cdot(\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$ σ_1, σ_2 zero: $D_{n1} = D_{n2}$ $\sigma_2 \to \infty$: $D_{n1} = \rho_s$
$\oint_S \mathbf{B}\cdot d\mathbf{s} = 0$	$\nabla\cdot \mathbf{B} = 0$	$B_{n1} = B_{n2}$ or $\mathbf{n}\cdot(\mathbf{B}_1 - \mathbf{B}_2) = 0$

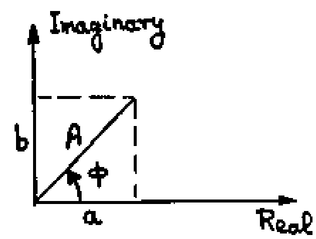
$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = \sigma \mathbf{E}, \quad \nabla\cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Complex Numbers and Phasor Technique

Complex Number : $\boxed{a + jb = A e^{j\phi} = A(\cos\phi + j\sin\phi)}$

where : $j = \sqrt{-1} \quad \Rightarrow j \cdot j = -1$

Then : $\boxed{\begin{array}{l} A = \sqrt{a^2+b^2} \\ \phi = \tan^{-1} \dfrac{b}{a} \end{array}}$



Note : $\cos\phi = \text{Re}(e^{j\phi})$

Time Harmonic Maxwell's Eqs. in Complex Form

$$\boxed{\begin{array}{l} \nabla \times E = -\dfrac{\partial B}{\partial t} \\ \nabla \times H = J + \dfrac{\partial D}{\partial t} \\ \nabla \cdot D = \rho \\ \nabla \cdot B = 0 \end{array}}$$

$E = E(\underbrace{u_1, u_2, u_3}_{\text{Space}}, \underbrace{t}_{\text{Time}})$

$H = H(u_1, u_2, u_3, t)$

$D = D(u_1, u_2, u_3, t)$

$B = B(u_1, u_2, u_3, t)$

$\vdots$

Introduce time dependence $e^{j\omega t}$ (Note $\cos\omega t = \text{Re}\, e^{j\omega t}$)

Then
$$\left.\begin{array}{l} E = \bar{E}(u_1, u_2, u_3) e^{j\omega t} \\ H = \bar{H}(u_1, u_2, u_3) e^{j\omega t} \\ D = \bar{D}(u_1, u_2, u_3) e^{j\omega t} \\ B = \bar{B}(u_1, u_2, u_3) e^{j\omega t} \\ \vdots \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} \nabla \times \bar{E} = -j\omega \bar{B} \\ \nabla \times \bar{H} = \bar{J} + j\omega \bar{D} \\ \nabla \cdot \bar{D} = \bar{\rho} \\ \nabla \cdot \bar{B} = 0 \end{array}}$$

Four Law's

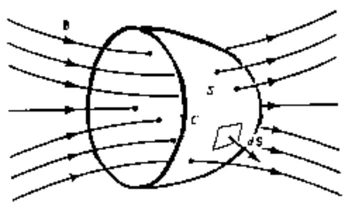
Maxwell's equations in integral form are a set of FOUR LAWS resulting from several experimental findings and a purely mathematical contribution.

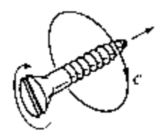
- Faraday's Law
- Ampere's Circuital Law
- Gauss' Law for the Electric Field
- Gauss' Law for the Magnetic Field

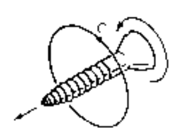
Faraday's Law

The electromotive force around a closed path is equal to the negative of the time rate of change of the magnetic Flux enclosed by the path

$$\oint_C E \cdot d\ell = -\frac{d}{dt}\int_S B \cdot ds$$







Ampere's Circuital Law

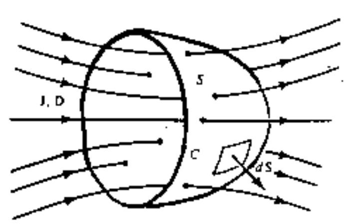
The magnetomotive force around a closed path is equal to the algebraic sum of the current due to flow of charges and the displacement current bounded by the path

Displacement Current introduced by Maxwell

$$\oint_C H \cdot d\ell = [I_c]_S + \frac{d}{dt}\int_S D \cdot ds$$

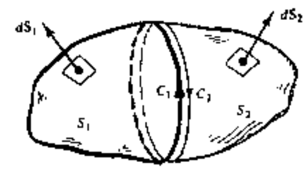
Current due to flow of free Charges

$$[I_c]_S = \int_S J \cdot ds$$



Application of Ampere's Law to Closed Surfaces

$$\oint_{C_1} H \cdot d\ell = \int_{S_1} J \cdot dS_1 + \frac{d}{dt}\int_{S_1} D \cdot dS_1$$



$$\oint_{C_2} H \cdot d\ell = \int_{S_1} J \cdot dS_2 + \frac{d}{dt}\int_{S_2} D \cdot dS_2$$

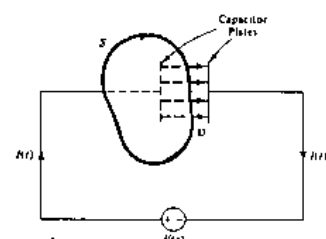
$$0 = \oint_{S_1+S_2} J \cdot dS + \frac{d}{dt}\int_{S_1+S_2} D \cdot dS$$

Finally: $\boxed{\frac{d}{dt}\oint_S D \cdot dS = -\oint_S J \cdot dS}$

Displacement current emanating from a closed surface is equal to the current due to charges flowing in the volume bounded by the closed surface.

Capacitor Circuit

$$\frac{d}{dt}\oint_S D \cdot dS = I(t)$$



Ignoring the fringe effects of finite plates

$$\boxed{\frac{d}{dt}(D \overset{\text{area of each plate}}{A}) = I(t)}$$

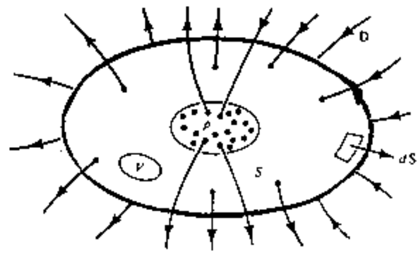
Observation: Where the wire current ends on one of the plates, the displacement current takes over and completes the circuit to the second plate.

Gauss' Law for Electric Field

The displacement flux emanating from a closed surface is equal to the charge contained within the volume The volume bounded by the surface

$$\oint_S D\cdot dS = [Q]_V \quad \text{free charge} \quad = \int_V \rho\, dv \quad \text{charge density}$$

$$\boxed{\oint_S D\cdot dS = \int_V \rho\, dv}$$

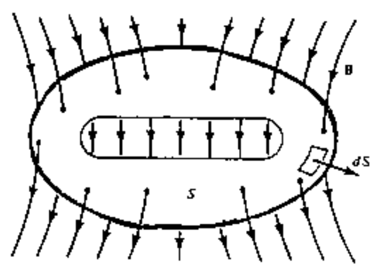


$D = \epsilon E$

Gauss' Law for Magnetic Field

The magnetic flux emanating from a closed surface is equal to zero

$$\boxed{\oint_S B\cdot dS = 0}$$

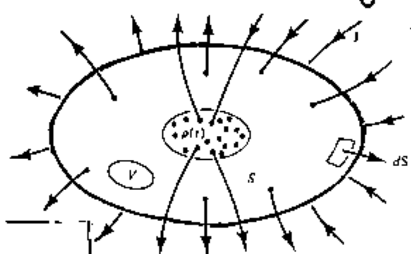


Note that Gauss' Law for magnetic field is consitent with Faraday's Law.

$$\left.\begin{aligned} \oint_{C_1} E\cdot dl &= -\frac{d}{dt}\int_{S_1} B\cdot dS_1 \\ \oint_{C_2} E\cdot dl &= -\frac{d}{dt}\int_{S_2} B\cdot dS_2 \end{aligned}\right] \Longrightarrow \quad 0 = -\frac{d}{dt}\oint_{S_1+S_2} B\cdot dS$$

Law of Conservation of Charge

The net current due to flow of charges emanating from a closed surface is equal to the time rate of decreases of the charge within the volume bounded by the surface



$$\boxed{\oint_S J \cdot ds = -\frac{d}{dt}\int_V \rho \, dv}$$

$$\oint_S D \cdot ds = \int_V \rho \, dv \qquad \text{Gauss' Law}$$

$$\frac{d}{dt}\oint_S D \cdot ds = -\oint_S J \cdot ds \qquad \text{Ampere's law}$$

Maxwell's Equations for Static Fields in Integral form

For static Fields $\frac{d}{dt}[\ \ ] = 0$

$$\oint_C E \cdot dl = 0$$

$$\oint_C H \cdot dl = \int_S J \cdot ds$$

$$\oint_S D \cdot ds = \int_V \rho \, dv$$

$$\oint_S B \cdot ds = 0$$

Note: The electric and magnetic fields are no longer dependent.

$$\oint_S J \cdot ds = 0$$

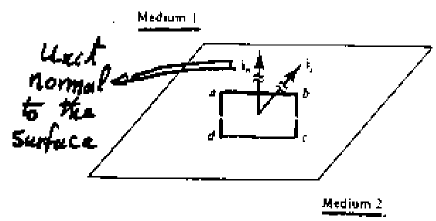
Boundary Conditions

Boundary conditions are a set of relationships relating the field components at a point adjacent to and on one side of boundary to the field components at a corresponding point adjacent to and on the other side of the boundary.

Boundary conditions are obtained using Maxwell's equations in Integral form.

Boundary Condition for $E_{tangential}$

$$\boxed{\begin{aligned} &\oint_c \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_s \mathbf{B}\cdot d\mathbf{S} \\ &\oint_c \mathbf{H}\cdot d\mathbf{l} = \int_s \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\int_s \mathbf{D}\cdot d\mathbf{S} \\ &\oint_s \mathbf{D}\cdot d\mathbf{S} = \int_v \rho\, dv \\ &\oint_s \mathbf{B}\cdot d\mathbf{S} = 0 \end{aligned}}$$



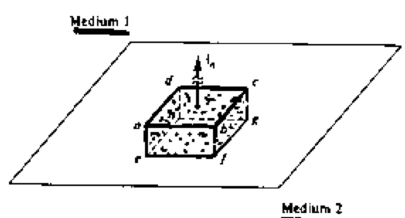
$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint \mathbf{E}\cdot d\mathbf{l} = -\lim_{\substack{ad\to 0\\ bc\to 0}} \frac{d}{dt}\int_{\substack{\text{area}\\ abcd}} \mathbf{B}\cdot d\mathbf{s} \longrightarrow 0$$

$$\underline{E_{ab}} + \underline{E_{cd}} = 0$$

$$\boxed{i_n \times (E_1 - E_2) = 0}$$

Tangential E fields are continous across the boundary.

Boundary Condition for D_{normal}



$$\lim_{SS\to 0} \oint_{\text{Surface}} \mathbf{D}\cdot d\mathbf{s} = \lim_{SS\to 0} \int_{\text{Volume}} \rho\, dv \longrightarrow 0 \text{ except for the surface charge}$$

dotted boundary

$$\underline{D_{n1}} - \underline{D_{n2}} = \rho_s$$

$$\boxed{i_n \cdot (D_1 - D_2) = \rho_s}$$

Normal components of D are discontinuous by the amount of surface charge density.

Boundary Condition for $H_{tangential}$

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint H\cdot dl = \lim_{\substack{ad\to 0\\ bc\to 0}} \int_{area} J\cdot ds + \lim_{\substack{ab\to 0\\ bc\to 0}} \frac{d}{dt}\int_{area} D\cdot ds$$

0 except for surface currents

0

$$\underline{H_{ab}} + \underline{H_{cd}} = (J_s \cdot i_s)$$

surface current flowing normal to the surface abcd

or

$$i_s \times i_n \cdot (H_1 - H_2) = J_s \cdot i_s$$

$$i_s \cdot i_n \times (H_1 - H_2) = J_s \cdot i_s \Rightarrow \boxed{i_n \times (H_1 - H_2) = J_s}$$

The tangential components of the H field are discontinuous by the amount of surface current

---

Boundary Condition for B_{normal}

$$\lim_{SS\to 0} \oint_{surface} B\cdot ds = 0$$

$$\underline{B_{n1}} - \underline{B_{n2}} = 0$$

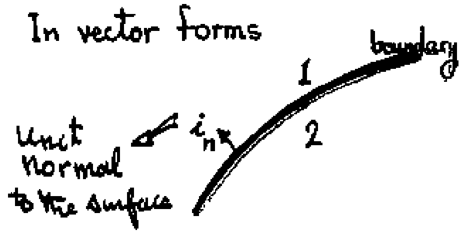
$$\boxed{i_n \cdot (B_1 - B_2) = 0}$$

Normal components of B are continuous.

Summary of Boundary Conditions

$$
\boxed{
\begin{aligned}
\mathbf{i}_n \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\
\mathbf{i}_n \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_S \\
\mathbf{i}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= \rho_S \\
\mathbf{i}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0
\end{aligned}
}
$$

In vector forms



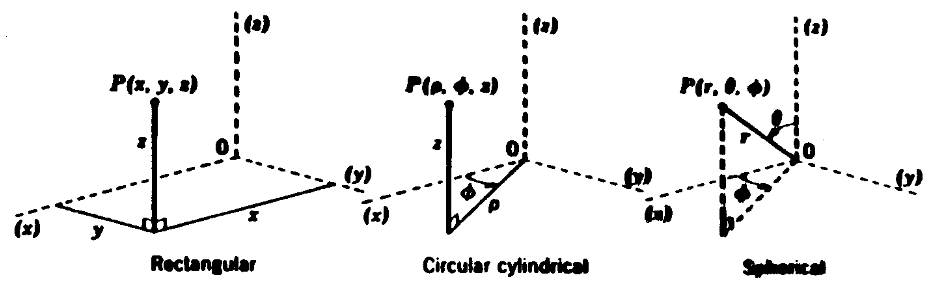
$$
\boxed{
\begin{aligned}
E_{t1} - E_{t2} &= 0 \\
H_{t1} - H_{t2} &= J_S \\
D_{n1} - D_{n2} &= \rho_S \\
B_{n1} - B_{n2} &= 0
\end{aligned}
}
$$

In scalar forms

Application of boundary conditions are essential in solving Maxwell's equations

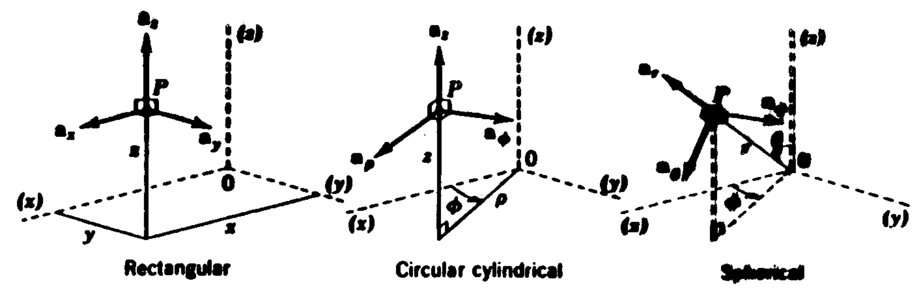
Coordinate Systems

Location of Point P:



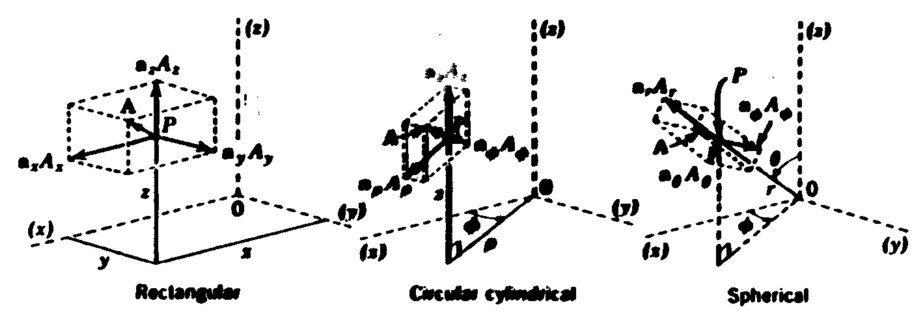
(a)

Unit vectors at P:



(b)

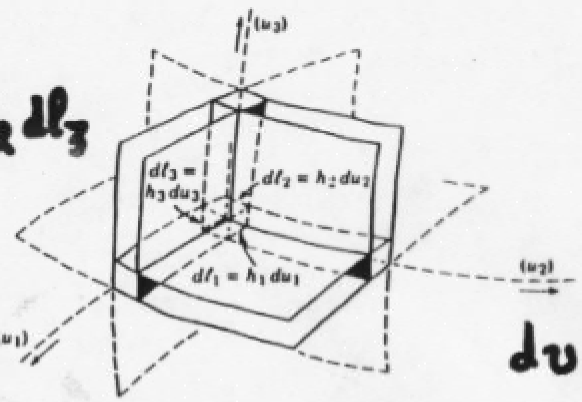
Resolution of vector A:



(c)

Differential Elements

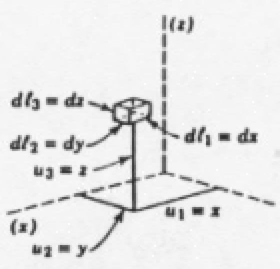
$$dv = dl_1\, dl_2\, dl_3$$



Generalized orthogonal coordinates

(a)

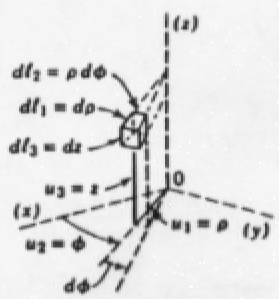
$$dv = h_1 h_2 h_3\, du_1\, du_2\, du_3$$



Rectangular

(b)

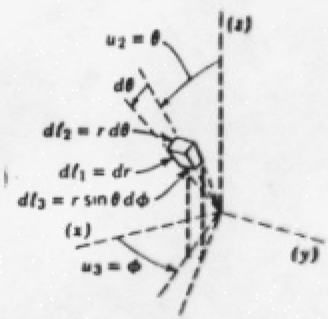
$h_1 = h_2 = h_3 = 1$



Circular cylindrical

(c)

$h_1 = 1$
$h_2 = \rho$
$h_3 = 1$

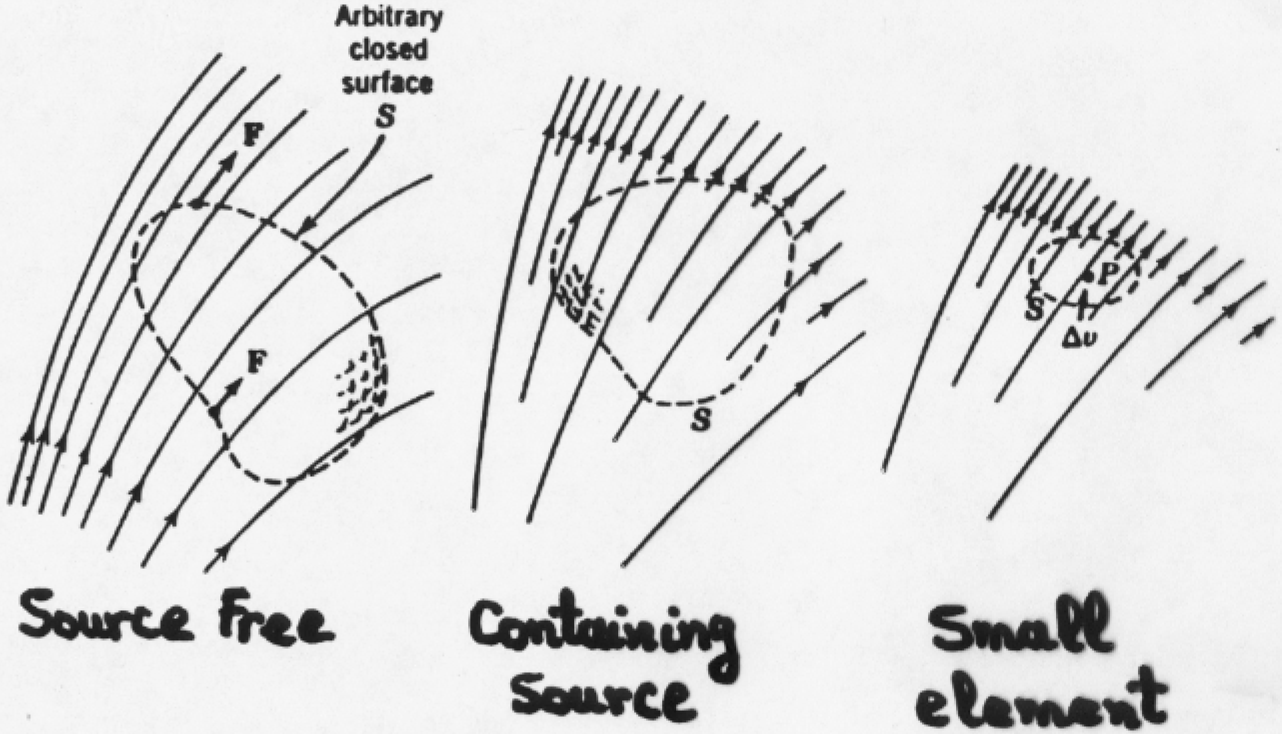


Spherical

(d)

$h_1 = 1$
$h_2 = r$
$h_3 = r \sin\theta$

Divergence of a Vector Function (1)



$$\text{div } \mathbf{F} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{F} \cdot d\mathbf{s}}{\Delta v} \text{ flux lines/m}^3$$

Divergence of a Vector Function [redacted] (2)

General:

$$\operatorname{div} \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right]$$

Rectangular:

$$\operatorname{div} \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Spherical:

$$\operatorname{div} \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

Cylindrical:

$$\operatorname{div} \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

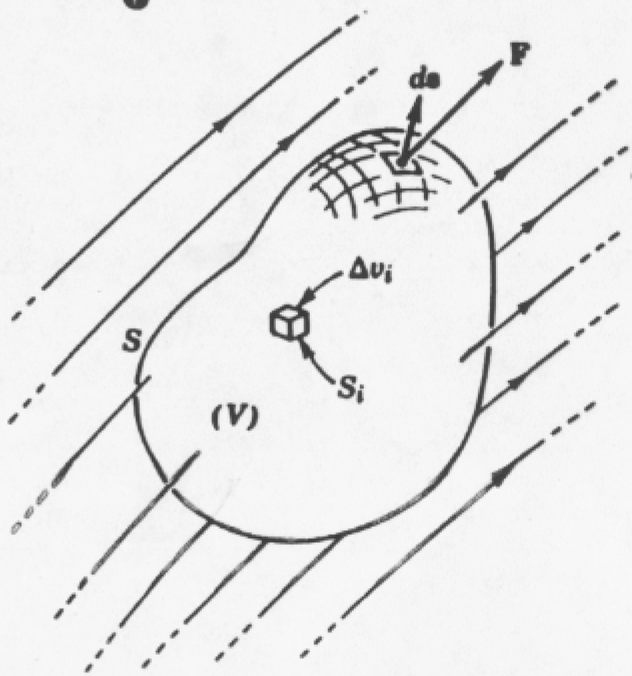
∇ operator:

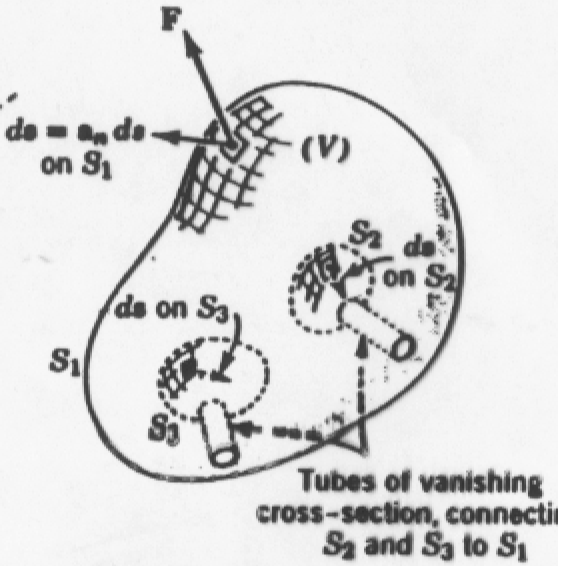
$$\nabla\cdot\mathbf{F} = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right)\cdot(\mathbf{a}_x F_x + \mathbf{a}_y F_y + \mathbf{a}_z F_z)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\boxed{\operatorname{div} \mathbf{F} \equiv \nabla\cdot\mathbf{F}}$$

Divergence Theorem





$$\oint_{S_i} \mathbf{F}\cdot d\mathbf{s} = (\operatorname{div}\mathbf{F})\,\Delta v_i$$

$$\sum_{i=1}^{n}\left[\oint_{S_i} \mathbf{F}\cdot d\mathbf{s}\right] = \oint_{S} \mathbf{F}\cdot d\mathbf{s}$$

$$\oint_{S} \mathbf{F}\cdot d\mathbf{s} = \lim_{\Delta v_i \to 0}\sum_{i=1}^{n} (\operatorname{div}\mathbf{F})\,dv = \int_V (\operatorname{div}\mathbf{F})\,dv$$

$$\boxed{\int_V (\operatorname{div}\mathbf{F})\,dv = \oint_S \mathbf{F}\cdot d\mathbf{s}}$$

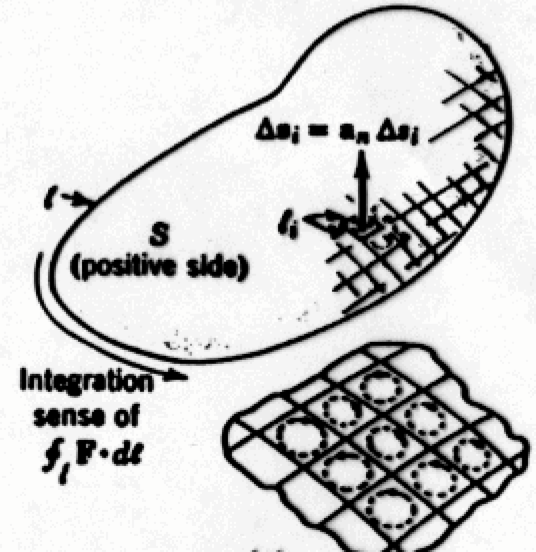
Stokes Theorem

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{s} = \oint_\ell \mathbf{F} \cdot d\boldsymbol{\ell}$$

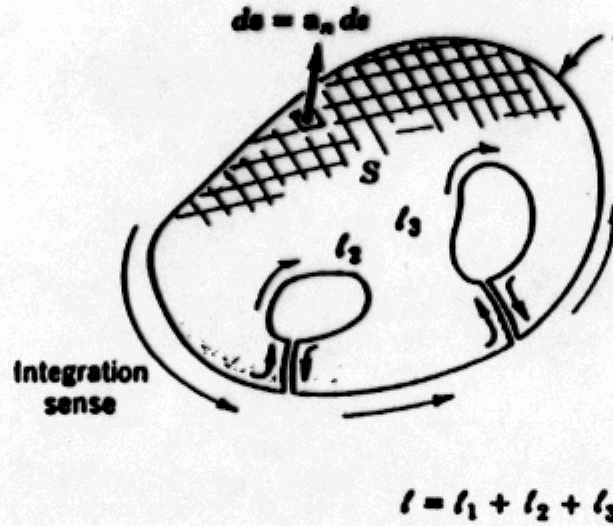
$$\oint_{\ell_i} \mathbf{F} \cdot d\boldsymbol{\ell} = [\text{curl } \mathbf{F}] \cdot \Delta \mathbf{s}_i$$

$$\sum_{i=1}^{n} \left[\oint_{\ell_i} \mathbf{F} \cdot d\boldsymbol{\ell} \right] = \oint_\ell \mathbf{F} \cdot d\boldsymbol{\ell}$$

$$\oint_\ell \mathbf{F} \cdot d\boldsymbol{\ell} = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} [(\text{curl } \mathbf{F}) \cdot \Delta \mathbf{s}_i] = \int_S (\text{curl } \mathbf{F}) \cdot d\mathbf{s}$$



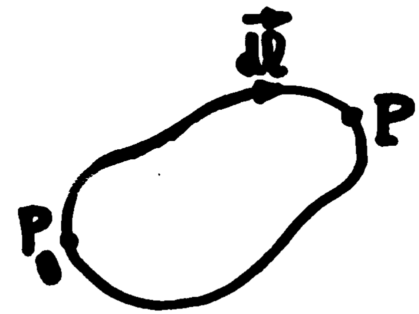
(a)



$\ell = \ell_1 + \ell_2 + \ell_3$

Integral Property of Gradient

$$\boxed{\oint_{l} (\operatorname{grad} f)\cdot d\mathbf{l} = 0}$$



Proof:

$$\int_{P_0}^{P} (\operatorname{grad} f)\cdot d\mathbf{l}$$

$$\int_{P_0}^{P} (\operatorname{grad} f)\cdot d\mathbf{l} = \int_{P_0}^{P} df = f\Big]_{P_0}^{P}$$
$$= f(u_1, u_2, u_3) - f(u_1^0, u_2^0, u_3^0)$$

Note: grad f is a conservative field.

Gradient of a Scalar Function (2)

Generalized Coordinates:

$$\text{grad}\, f \equiv \mathbf{a}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \mathbf{a}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \mathbf{a}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$|\text{grad}\, f| = \left[\left(\frac{\partial f}{h_1\,\partial u_1}\right)^2 + \left(\frac{\partial f}{h_2\,\partial u_2}\right)^2 + \left(\frac{\partial f}{h_3\,\partial u_3}\right)^2\right]^{1/2}$$

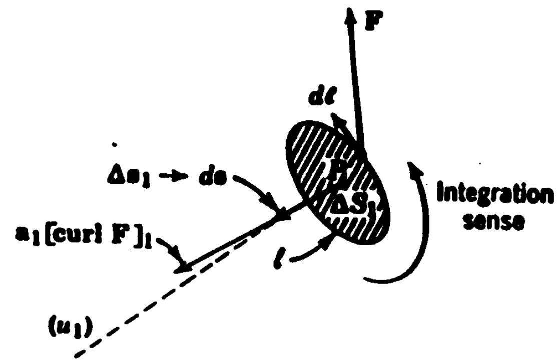
Rectangular:

$$\text{grad}\, f = \mathbf{a}_x \frac{\partial f}{\partial x} + \mathbf{a}_y \frac{\partial f}{\partial y} + \mathbf{a}_z \frac{\partial f}{\partial z}$$

Spherical:

$$\text{grad}\, f = \mathbf{a}_r \frac{\partial f}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Curl of a Vector Field (1)



$$\text{curl } \mathbf{F} = \mathbf{a}_1[\text{curl } \mathbf{F}]_1 + \mathbf{a}_2[\text{curl } \mathbf{F}]_2 + \mathbf{a}_3[\text{curl } \mathbf{F}]_3$$

$$\mathbf{a}_1[\text{curl } \mathbf{F}]_1 \equiv \mathbf{a}_1 \lim_{\Delta s_1 \to 0} \frac{\oint_l \mathbf{F} \cdot d\boldsymbol{l}}{\Delta s_1}$$

$$\text{curl } \mathbf{F} = \mathbf{a}_1 \lim_{\Delta s_1 \to 0} \frac{\oint_l \mathbf{F} \cdot d\boldsymbol{l}}{\Delta s_1} + \mathbf{a}_2 \lim_{\Delta s_2 \to 0} \frac{\oint_l \mathbf{F} \cdot d\boldsymbol{l}}{\Delta s_2} + \mathbf{a}_3 \lim_{\Delta s_3 \to 0} \frac{\oint_l \mathbf{F} \cdot d\boldsymbol{l}}{\Delta s_3}$$

Curl of a Vector Field [redacted] (2)

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{a}_1}{h_2 h_3} & \dfrac{\mathbf{a}_2}{h_3 h_1} & \dfrac{\mathbf{a}_3}{h_1 h_2} \\[2ex] \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\[2ex] h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\[1ex] \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\[1ex] F_x & F_y & F_z \end{vmatrix}$$

$$\operatorname{curl} \mathbf{F} \equiv \nabla \times \mathbf{F}$$

$$\operatorname{curl} \mathbf{F} = \frac{\mathbf{a}_r}{r \sin\theta}\left[\frac{\partial}{\partial \theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right] + \frac{\mathbf{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}(rF_\phi)\right] + \frac{\mathbf{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]$$

$$\operatorname{curl} \mathbf{F} = \mathbf{a}_\rho\left[\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right] + \mathbf{a}_\phi\left[\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right] + \mathbf{a}_z\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\phi) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right]$$

Laplacian Operator (I)

$$\nabla f = \mathbf{a}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \mathbf{a}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \mathbf{a}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\nabla \cdot (\nabla f) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

$$\nabla \cdot (\nabla f) \equiv \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian Operator (2)

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$$

General:

$$\nabla^2 = \nabla \cdot \nabla = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]$$

Spherical:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Cylindrical:

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian Operator (3)

General:

$$\nabla^2 \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right](\mathbf{a}_1 F_1 + \mathbf{a}_2 F_2 + \mathbf{a}_3 F_3)$$

Rectangular:

$$\nabla^2 \mathbf{F} = \mathbf{a}_x \nabla^2 F_x + \mathbf{a}_y \nabla^2 F_y + \mathbf{a}_z \nabla^2 F_z$$

Cylindrical:

$$\nabla^2 \mathbf{F} = \mathbf{a}_\rho\left[\nabla^2 F_\rho - \frac{2}{\rho^2}\frac{\partial F_\phi}{\partial \phi} - \frac{F_\rho}{\rho^2}\right] + \mathbf{a}_\phi\left[\nabla^2 F_\phi + \frac{2}{\rho^2}\frac{\partial F_\rho}{\partial \phi} - \frac{F_\phi}{\rho^2}\right] + \mathbf{a}_z \nabla^2 F_z$$

Curl x Curl x Operator (2)

$$\nabla(\nabla\cdot\mathbf{F}) = \mathbf{a}_x\frac{\partial^2 F_x}{\partial x^2} + \mathbf{a}_y\frac{\partial^2 F_y}{\partial y^2} + \mathbf{a}_z\frac{\partial^2 F_z}{\partial z^2} + \mathbf{a}_x\left[\frac{\partial}{\partial x}\left(\frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\right]$$
$$+ \mathbf{a}_y\left[\frac{\partial}{\partial y}\left(\frac{\partial F_z}{\partial z} + \frac{\partial F_x}{\partial x}\right)\right] + \mathbf{a}_z\left[\frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)\right]$$

$$\nabla(\nabla\cdot\mathbf{F}) = \mathbf{a}_x\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}\right) + \mathbf{a}_y\left(\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2}\right)$$
$$+ \mathbf{a}_z\left(\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}\right)$$
$$+ \mathbf{a}_x\left[\frac{\partial}{\partial x}\left(\frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2}\right]$$
$$+ \mathbf{a}_y\left[\frac{\partial}{\partial y}\left(\frac{\partial F_z}{\partial z} + \frac{\partial F_x}{\partial x}\right) - \frac{\partial^2 F_y}{\partial z^2} - \frac{\partial^2 F_y}{\partial x^2}\right]$$
$$+ \mathbf{a}_z\left[\frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) - \frac{\partial^2 F_z}{\partial x^2} - \frac{\partial^2 F_z}{\partial y^2}\right]$$

Important vector identity:

$$\star \quad \nabla^2\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F}) \quad \star$$

☆ Summary of Vector Identities ☆

Algebraic

(1) $\mathbf{F}\cdot\mathbf{G} = \mathbf{G}\cdot\mathbf{F}$

(2) $\mathbf{F}\times\mathbf{G} = -\,\mathbf{G}\times\mathbf{F}$

(3) $\mathbf{F}\cdot(\mathbf{G}+\mathbf{H}) = \mathbf{F}\cdot\mathbf{G} + \mathbf{F}\cdot\mathbf{H}$

(4) $\mathbf{F}\times(\mathbf{G}+\mathbf{H}) = \mathbf{F}\times\mathbf{G} + \mathbf{F}\times\mathbf{H}$

(5) $\mathbf{F}\times(\mathbf{G}\times\mathbf{H}) = \mathbf{G}(\mathbf{H}\cdot\mathbf{F}) - \mathbf{H}(\mathbf{F}\cdot\mathbf{G})$

(6) $\mathbf{F}\cdot(\mathbf{G}\times\mathbf{H}) = \mathbf{G}\cdot(\mathbf{H}\times\mathbf{F}) = \mathbf{H}\cdot(\mathbf{F}\times\mathbf{G})$

Integral

(7) $\oint_s \mathbf{F}\cdot d\mathbf{s} = \int_v \nabla\cdot\mathbf{F}\, dv$

(8) $\oint_\ell \mathbf{F}\cdot d\boldsymbol{\ell} = \int_s (\nabla\times\mathbf{F})\cdot d\mathbf{s}$

(9) $\oint_s f(\nabla g)\cdot d\mathbf{s} = \int_v [f\nabla^2 g + (\nabla f)\cdot(\nabla g)]\, dv$

(10) $\oint_s [f\nabla g - g\nabla f]\cdot d\mathbf{s} = \int_v (f\nabla^2 g - g\nabla^2 f)\, dv$

Differential

(11) $\nabla(f+g) = \nabla f + \nabla g$

(12) $\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$

(13) $\nabla\times(\mathbf{F}+\mathbf{G}) = \nabla\times\mathbf{F} + \nabla\times\mathbf{G}$

(14) $\nabla(fg) = f\nabla g + g\nabla f$

(15) $\nabla\cdot(f\mathbf{F}) = \mathbf{F}\cdot\nabla f + f(\nabla\cdot\mathbf{F})$

(16) $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$

(17) $\nabla\times(f\mathbf{F}) = (\nabla f)\times\mathbf{F} + f(\nabla\times\mathbf{F})$

(18) $\nabla\cdot\nabla f = \nabla^2 f$

(19) $\nabla\cdot(\nabla\times\mathbf{F}) = 0$

(20) $\nabla\times(\nabla f) = 0$

(21) $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$

(22) $\nabla\times(f\nabla g) = \nabla f\times\nabla g$